INEFFICIENT INFORMATION
AGGREGATION AS A SOURCE OF ASSET
PRICE BUBBLES

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ABSTRACT

This paper presents a new theory of bubbles, or discrepancies between the market clearing price and the fundamental value of an asset. In our setting, Bayesian traders, oriented towards long-term gains, receive private information ('news') and also make inferences from noisy price signals. Price exhibits higher variance than fundamental value (the latter defined as fully-aggregated expected value) especially when news is informative but infrequent. The corresponding bubbles are self-limiting but may exhibit momentum and overshooting. A parametric example, involving the exponential/gamma conjugate families, is provided.

We don’t have any penetrating explanations of yesterday’s stock market, but we certainly believe that stocks do not fall 86½ points for nothing.

The general case for a drop in the market after its recent record highs is clear enough. The Fed..., ..., the tax bill.

None of this, though, was any different on Thursday than it was with the market at its peak six sessions ago. News...and rumors yesterday...were certainly negative but scarcely dramatic.

Some market pros believe this kind of a drop is merely the market catching up with what it already knew. We doubt it. Our hunch is that something changed between Wednesday and Thursday, and that eventually we’ll learn what it was (Wall Street Journal Editorial, Friday, September 12, 1986).

1. INTRODUCTION

Not all financial economists would subscribe to this touching confession of faith in the strong-form Efficient Markets Hypothesis, but it is hard to

*We would like to thank economics and finance seminar participants at the University of Arizona, University of California-Berkeley, University of California-Santa Cruz, and VPI for useful comments, and Shawn LaMaster for excellent programming assistance. We are indebted also to three referees for very useful editorial suggestions.
come by intellectually respectable models of 'the market catching up with what it already knew'. Our principal goal in this paper is to provide such a model, one that is consistent with semi-strong efficient markets and trader rationality but that leaves room for at least moderate 'bubbles' in asset prices arising from imperfect aggregation of private information. To provide a context for our ideas, we begin with a brief review of previous approaches to bubbles; i.e., discrepancies between an asset's market price and its fundamental value.

Many of the oldest studies of the business cycle focused on asset price instability, and posited that some collective 'mania' occasionally caused investors to bid up asset prices to unsustainable levels, eventually but inevitably ending in a 'panic' as prices crashed. See Kindleberger (1978) for a modern summary of this so-called 'lemmings' literature and a discussion of dozens of major historical bubbles.

Keynes (1936) in his famous 'beauty contest' metaphor pointed out that asset traders who attempt to profit from short-run price fluctuations must rationally compare price with their expectations of others' expectations, rather than their own estimates of fundamental values, so perhaps investor behavior is not entirely irrational. Several modern writers have related this Keynesian theme to the multiplicity of rational expectations equilibria in many models. Azariadis (1981), for example, suggests that bubbles should be thought of as instantaneous transitions, triggered by extraneous events, between different 'self-fulfilling prophecy' (or 'sunspot') equilibria. The perfect coordination of expectations (and actions) in this approach, however, seems quite at odds with the turmoil and confusion generally associated with historical bubbles.

Perhaps the most popular approach in recent years is based on the observation (originally due to Hahn (1966)) that no-intertemporal-arbitrage conditions do not yield a unique price path in most perfect foresight or rational expectations equilibrium asset-market models. Typically one has a convergent saddle-path identified as the 'fundamental', as well as other price paths that diverge from the saddle path at an exponential rate. Such divergent paths can not always be eliminated by transversality conditions, particularly in stochastic versions. For instance, Blanchard and Watson (1983) give examples of such bubbles with random lifetimes that grow at a known exponential rate (the discount rate plus a risk premium) until they burst. The information conditions in models of this type are seldom spelled out, but it is hard to avoid the interpretation that once the bubble has started everyone knows that price is above fundamental value and therefore the game has an expected negative sum for current and future transactors in the asset market, as in a Ponzi scheme or chain letter. Tirole (1982) points out that such bubbles are impossible in Rational Expectations Equilibrium once the negative sum aspect is common knowledge, except for special cases in which the losses can be passed on forever to later participants. Nevertheless, this 'exponential rational bubbles' approach generated numerous articles.
Guth (1984) presents an alternative rational expectations view of bubbles. He posits an incomplete information game between ordinary speculators, better-informed speculators, and liquidity-motivated transactors, and shows that in perfect Bayesian–Nash equilibrium the fundamental value (based on the aggregate information) can differ with positive probability from the transacted price. The better-informed speculators (a non-negligible fraction of the market) must be able to form an effective cartel for such a bubble to arise. Such an 'information-monopoly bubble' reminds one of some popular accounts of the Hunt brothers' silver market activities in the late 1970s. Summers and his coauthors (e.g., DeLong et al. (1990)) recently introduced a new approach in which some irrational traders create excess volatility which rational traders cannot eliminate by arbitrage. Indeed, rational speculation can actually reinforce discrepancies between fundamental value and price created by irrational 'noise' or 'positive feedback' traders.

Our own model also emphasizes heterogeneous beliefs, but we do not assume behaviorally distinct types of traders. In the spirit of Hirshleifer's (1989) critique of Tirole, we study a dynamic market in which traders receive heterogeneous private information ('news'), trade, observe price and receive more 'news', trade again, etc., over many periods. Our focus is the extent to which price aggregates the diverse news. Under quite general specifications regarding expectation formation we show that aggregation is imperfect and substantial discrepancies can arise between price and fundamental value.

We take some care to model the market institution through which traders' decisions yield transactions and observed prices, since this institution largely determines the extent and timing of the public information conveyed by prices. We employ a simplified version of the Clearinghouse institution (sometimes referred to in the literature as a Call market or a sealed bid-offer auction) because it is widely used in practice (Schwartz, 1988), analytically tractable, and bears some resemblance to the Walrasian institution that is usually employed in theoretical discussions (but almost never used in practice).\(^1\)

In other respects, we keep our model as rudimentary as possible. To distinguish our model clearly from its predecessors, we employ the following simplifications: (1) traders pursue buy-and-hold strategies, eliminating 'beauty contests' bubbles; (2) traders neglect the possibility that they may affect prices, eliminating 'information monopoly' bubbles; and (3) traders are otherwise rational wealth-maximizers, eliminating the most obvious 'lemmings bubbles'.

\(^1\) In our view, the Walrasian auctioneer institution is inappropriate for serious modeling of information aggregation despite its widespread use for this purpose. For example, in his classic rational expectations equilibrium (REE) model Grossman (1976) assumes that traders are able to observe equilibrium prices before submitting their demand schedules to a Walrasian auctioneer, a logistical impossibility. (See also the critical comments of Allan Kraus following the Grossman article.) Here we work with an explicit, feasible market institution and seek updating rules that can be applied even outside of equilibrium.
A final modeling choice deserves brief discussion. It is convenient to avoid specifying risk preferences because they do not play a central role in our view of information aggregation. However, a risk-neutral agent will want to take an arbitrarily large position if he perceives even a small price discrepancy, so difficulties arise when agents' perceptions differ. We avoid the problem by imposing arbitrary limits on position sizes — a choice that we feel (given bankruptcy costs, impediments to short-selling, etc.) does not do too much violence to current practice.

In the next section we introduce the market structure and present a parametric example drawn from our 1986 paper in order to build intuition and sharpen the issues. We begin the following section with a rather general formal specification of private information and Bayesian updating processes, and then state several analytical results.

We derive a formula for the optimal trader action and investigate its dependence on past prices in the first two Propositions. Then we characterize the market clearing price and obtain relations between this price, the fundamental, and the unobservable 'true value' of the asset. The rest of our results concern the dynamic behavior of bubbles: we find that they can be 'self-feeding' in that once started, small positive (or negative) bubbles tend to grow (Proposition 6) but that they eventually are self-correcting in that massive positive (or negative) bubbles tend to shrink (Proposition 5). We also discuss the comparative statics of the information arrival process, and argue that, other things equal, bubbles tend to be larger when information is 'lumpier', i.e., less frequent but more precise. Since news is likely to be lumpier in this sense for a smaller company than for a large diversified company, this result suggests that a small company mutual fund has unusual profit opportunities to buy at the bottom of a negative bubble and to sell at the top of a positive bubble (and also greater risks of doing the opposite). Of course, it is central to our analysis that no trader really knows whether there is a bubble, but each trader can make his own estimate.

The final section summarizes our results and discusses empirical implications and the relevant empirical literature. The Appendix contains the proofs and some calculations.

2. THE ASSET MARKET: AN EXAMPLE

At some given time $T$ in the future, each of $M$ indivisible shares in some venture\textsuperscript{2} will pay off $S_X$. We assume here that it is common knowledge that

\textsuperscript{2} Approximate counterparts of such 'ventures' in contemporary financial markets include European call options, takeovers in which the firm is to be liquidated at a specified date, and the conversion of a closed end mutual fund to an open end fund at a specified date. Theorists' 'state-contingent claims' also fall into this class. See Glosten and Milgrom (1984) for a tradition-based justification of assets similar to our 'venture'.
$X$ is an exponentially distributed random variable with mean $1/\lambda$. The true value $\hat{\lambda}$ of $\lambda$ is unknown.

There risk-neutral traders can exchange their shares for cash in a simple Clearinghouse market (rules specified below) at times $t = 1, 2, \ldots, T - 1$. We assume that traders pursue buy-and-hold strategies oriented to period $T$ wealth, and that they neglect the possibility that they can influence prices. We also assume a constant discount rate $r$ and exogenous limits on traders’ position sizes; for simplicity we set $r = 0$ and set the lower position limit at 0 (no short sales) and the upper limit at 1. For present purposes we make the convenient assumption that there are $N = 2M - 1$ traders, so there are $M$ shareholders and $M - 1$ non-shareholders.

Our simplified Clearinghouse works as follows. At the end of each period $t$, each trader privately submits a single ‘limit order’ $v_{it} \geq 0$, sometimes referred to as a ‘bid’. For a shareholding trader $i$ this order is an offer to sell his share at any price exceeding $v_{it}$, while for a non-shareholding trader the order is an offer to buy at any price not exceeding $v_{it}$.

Thus one obtains the supply curve $S_i(p)$ by arranging shareholders’ orders in ascending order and the demand curve $D_i(p)$ by arranging non-shareholders’ orders in descending order. See Figure 1 for an illustration. The clearing price $p_{t+1}$ is set at the (highest) intersection of $D_i$ and $S_i$, viz., $p_{t+1} = \max\{p : S_i(p) \leq D_i(p)\}$, and is announced (if any transactions occurred) at the beginning of period $t + 1$. The corresponding orders are executed at the same time, so (apart from ties) the $M$ highest bidders are the shareholders at time $t + 1$ and the clearing price is always the $M$th highest (here the median) bid.

The focus on our analysis is the information flow. We assume that in each period each participant costlessly receives, with known probability $p > 0$, some private information ('news'), denoted $z_{it}$. In our parametric example, news is an independent realization from a Gamma distribution with mean $a/b$ and variance $a/b^2$, where $a > 1$ is a known positive constant and $b = \lambda \bar{a}$. This distribution is quite convenient because its population mean is $\bar{a}/(\lambda \bar{a}) = 1/\lambda = E(X | \lambda) = \bar{X}$, the 'true' expected payoff, and its variance is proportional to the known parameter $\bar{a}$.

The impact of such news can be computed in a straightforward manner using basic properties of the Gamma distribution (see our working paper or DeGroot (1970), for example). Suppose prior beliefs regarding payoffs can be summarized in the (conjugate) Gamma distribution $g(\lambda | a, b)$ with $a > 1$ and $b > 0$. Then a news message $z = z_{it}$ induces posterior beliefs which are also Gamma with parameters $a + \bar{a}$ and $b + \bar{a}z$, i.e., the posterior

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1 Satterthwaite and Williams (1989) show that in theory the ability to influence price in a Clearinghouse market is negligible even with only a few traders. Friedman and Ostroy (1991) find strong empirical support for this claim.

4 The requirement that all traders submit limit orders is not restrictive. A seller can in effect submit a market order by specifying $v = 0$, and can effectively withdraw from the market by specifying $v$ at an absurdly high level; and analogously for buyers.
distribution is \( g(\lambda | a + \hat{a}, b + \hat{a}z) \). It follows that the prior expected payoff is \( x^0 = b / (a - 1) \) and the posterior expected payoff is \( \hat{x} = (b + \hat{a}z) / (a + \hat{a} - 1) \).

We also must consider the impact of publicly observed transactions prices \( p_t \). Since neither these prices nor their first differences \( \Delta p_t \) generally have a Gamma distribution, there is no tractable formula for the precise impact of such public information. However, it is reasonable to suppose that the main impact of an observed price change is a proportional change in the expected payoff, and that effects on the precision parameter \( (a) \) are
inefficient information aggregation

Negligible. To extend the present example we therefore assume that beliefs summarized in the Gamma distribution \( g(\lambda | a, b) \) are modified to \( g(\lambda | a, b + c\Delta p) \) after the price change \( \Delta p \) is observed, for some \( c \geq 0 \). This formula can be justified as a first-order approximation in \( \Delta p \), with the parameter \( c \) depending (inversely) on the amount of exogenous 'noise' (i.e., liquidity trading) in the market. The cumulative impact of public information \( \{p_0, p_1, \ldots, p_t\} \) on beliefs in this formulation is therefore a shift in the \( b \) parameter by

\[
\sum_{s=1}^{t} c\Delta p_s = c(p_t - p_0).
\]

We now are prepared to analyze trader behavior and market outcomes. Suppose a trader (index \( i \) suppressed in the next two paragraphs) enters period \( t \) with beliefs \( g(\lambda | a_{t-1}, b_{t-1}) \), observes the price \( p_t \) (generated from period \( t-1 \) orders) and receives news \( z_t \). Then his new beliefs are \( g(\lambda | a_t, b_t) \) where \( a_t = a_{t-1} + \hat{a} \) and \( b_t = b_{t-1} + \hat{a}z_t + c(p_t - p_{t-1}) \). At first one might suppose that our buy-and-hold assumption leads to an optimal order price of \( Y = \hat{X} = E(\lambda | a_t, b_t) = \frac{b_t}{a_t - 1} \), but further reflection discloses a variant of the 'winner's curse' problem: generally, a price change will be required for the order to be executed. If a trader does not condition his order on this prospective price change then he often will pay more (or receive less) for the asset than he is \textit{ex post} willing.

Thus a prospective buyer (a non-shareholding trader) optimally chooses his bid \( v_t \) by solving

\[
\max_{v \geq 0} E(X - p_{t+1} | [p_{t+1} \geq v], a_t, b_t, p_t) \Pr[p_{t+1} \geq v],
\]

i.e., maximizing the expected gain \( (X - p_{t+1}) \) conditioned on the event \( [p_{t+1} \geq v] \) that the bid will be executed and conditioned on current information \( (a_t, b_t, p_t) \). The first-order condition for this problem is

\[5\] Although \( c \) can also depend on the time period and/or the current parameter values \( a \) and \( b \), e.g., in rational expectations equilibrium, in this section we will treat it as a constant for simplicity. Assumption (B3) in Section 3 below is much less restrictive.

Our 1985 working paper explicitly models 'noise traders' whose activity is driven by exogenous liquidity shocks. For instance, a trader may sell because he has an unanticipated need for cash. Therefore, his sale does not necessarily imply he received new unfavorable information, and there is a 'surplus' available for other traders. Absent such noise, only an individual who believed he had superior information would attempt to transact, and therefore transactions would not occur in equilibrium, as Tirole has pointed out. We were not able to obtain any further insights by explicitly considering liquidity shocks and so we omit them in the present model in order to keep our notation and statement of results less cumbersome. However, liquidity shocks remain implicit in the present model as an explanation of traders' willingness to trade and as the underlying determinant of the parameter \( c \), or more generally, of the price response sensitivity \( \phi \), specified in assumption (B3) below.

\(v = E(X|a, b, \Delta p_{t+1} = v - p_t) = (b + c(v - p_t))/(a - 1)\), and it follows (restoring subscripts) that the optimal bid is \(v_i = (b_i - cp_i)/(a_i - c - 1)\). A similar calculation for a prospective seller (a shareholding trader) yields the same formula. Note that \(v_i \leq \bar{x}_{i} = b_i/(a_i - 1)\) as \(\bar{x}_{i} \leq p_i\) for \(c > 0\), as a result of Lemma 1 of the Appendix. In particular, bids and expectations have the same ordering over traders \(i = 1, \ldots, N\).

By the Clearinghouse rules, the price \(p_{t+1}\) will be the median bid \(v_{(M)}\), and traders with higher bids will be shareholders at \(t + 1\) (having purchased or retained their shares) while those with lower bids will be non-shareholders (having sold or failed to purchase shares). The expectations \(\bar{x}^0_{t+1}\) at the beginning of period \(t + 1\) (with \(p_{t+1}\) having been announced) will have the same ordering, i.e., shareholders more optimistic (higher \(\bar{x}^0\)) than non-shareholders, but as news arrives during the period the ordering of expectations may be altered and transactions (announced at the beginning of period \(t + 2\)) may result.

The price \(p_{t+1}\) generated by the Clearinghouse can usefully be compared to the fundamental value \(f_{t+1}\), the expected payoff conditioned on all information received by market participants up to time \(t\). If news messages \(\{z_1, \ldots, z_n\}\) have been received by participants in periods \(\{1, \ldots, t\}\), then

\[
f_{t+1} = (1/n) \sum_{j=1}^{n} z_j,
\]

the sample mean of news.

We ran a series of Monte Carlo simulations of this parametric example to compare price to fundamental. A typical simulation appears in Figure 2; the ‘true value’ of the expected payoff \(\bar{X} = 1/\lambda\) is normalized to 1.00, and traders’ time 0 priors for \(\lambda\) all have \(a_{i0} = 2\) and \(b_{i0} = 1\) so they are unbiased. There are \(M = 5\) shares and \(N = 9\) traders, each with price-sensitivity \(c = 0.5\). News arrives with probability \(\pi = 0.5\) and has precision parameter \(\tilde{a} = 1\). As one can see, the fundamental initially jumps to over 1.60 at \(t = 2\), but appears to converge to the ‘true value’ 1.00 by the end of the simulation at \(t = 20\). The price appears to take longer to settle down and appears to be weakly correlated with the fundamental.

Perhaps the first question one might wish to ask is whether (or to what extent) the market price \(p_t\) is more volatile than the fundamental value \(f_t\). An answer is difficult to glean from a time graph since in our simple example both converge to the ‘true value’ \(X\). Table 1 addresses the question by computing the ratio \(EV_t\) of the (cross-sectional) variance of \(p_t\) to the variance of \(f_t\) in each time period over 30 Monte-Carlo simulations. In the first column, parameters are the same as in Figure 2 except that the market is about twice as large (\(N = 19\) traders, \(M = 0\) shares) and so is the price sensitivity at \(c = 1.0\). In the other columns the probability \(\pi\) of receiving news is reduced as indicated, with compensating increases in
news precision. The $\bar{EV}_t$ statistics seem to rise\(^7\) as news becomes ‘lumpier’ (i.e., less frequent but more precise), and that (except in the first few periods) the price seems considerably (typically 2–3 times) more volatile than the fundamental.

A very interesting set of questions arise concerning the co-movements of price and fundamental. Defining the bubble as the difference between the two, i.e. $b_t = p_t - f_t$, one might ask whether prices merely follow a random walk or, more interestingly, whether the bubbles tend to feed on themselves (e.g., positive autocorrelation) and/or tend to burst (e.g., larger $|b_t|$ implies higher probability that $|b_{t+1}| < |b_t|$). For reasons to be explained in the next section we investigate these possibilities by replacing $c$ in the bid formula (but not in the updating formula) by $dc$ where the ‘hypothetical discount factor’ $d$ is between 0 and 1.

The simulation displayed in Figure 3 employs the same parameters as that in Figure 2 except that $d = 0.5$ rather than 1.0. One can imagine that the positive bubble $(p_t - f_t)$ at $t = 2$ feeds on itself before collapsing at $t = 8$, overshooting slightly, reversing itself and then feeding on itself again during periods 10–19 (bearing in mind the underlying trend for $b_t \to 0$ since $p_t$ and $f_t$ both converge to the ‘true value’).

\(^7\)The main exception is for $t = 17–20$ when $\pi = 0.2$. The corresponding price variances happened to be low in this sample of 30 runs, depressing $\bar{EV}_t$. An explanation of the consistently low $\bar{EV}_t$ for small $t$ is that for a small sample of news messages with a long-tailed distribution (Gamma), the median is less sensitive to outliers than the mean.
### TABLE 1

*Excess Variance Ratios, EV, = Var p, / Var f,*

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**Notes:** $\hat{a} = 1/\pi$. Other parameters are constant at $N = 19$, $M = 10$, $a_0 = 2.0$, $b_0 = 1.0$, $c = 1.0$, $HDF = 1.0$, and $\lambda = 1.0$. Each column of estimates is based on 30 Monte-Carlo simulations, across which $\text{Var } p$, and $\text{Var } f$, were computed for each $t$.

These simulations are meant only to be suggestive. In the next section we present a more formal analysis of excess volatility, overshooting, etc. in the context of a much more general information environment.

3. **THE FORMAL MODEL**

Our main interests are in the processes of belief adjustment and information aggregation. In order to model these processes in a general but tractable manner, we minimize the complexity of other parts of the model. Thus we suppose that $M$ indivisible shares are traded in a simple Clearing-house market by $N > M$ risk-neutral traders. Each share (a maximum of one per customer) pays a liquidating dividend of $X$ at a known time $T$. Subject to the single share constraint, traders buy and sell at times $t = 1, 2, \ldots, T$.
so as to maximize current expectation of final wealth, \( E_i(X - p) \) for a buyer and \( E_i(p_i - X) \) for a seller.\(^8\)

The payoff \( X \) is uncertain and its distribution is not known, but traders receive information which improves their estimates of the distribution. We begin to formalize the belief adjustment process by assuming that there is an indexed family of possible distributions \( F(X; \theta) \) for \( X \), and beliefs are expressed in terms of the index \( \theta \). In the previous section, for example, we assumed that \( F \) was the family of exponential distributions with index \( \theta = \lambda \). Two other possibilities are that \( F \) is binomial (so \( X = 0 \) or 1, as in an Arrow security) and the index \( \theta \) is the probability \( p \) that \( X = 1 \), or that \( \ln X \) is Normal with index vector \( \theta = (\mu, \sigma) \). For our general model it does no harm to assume that \( \theta \) is a point in a subset \( \Theta \) of \( \mathbb{R}^n \), but even more general specifications can easily be accommodated.

We take the very general view that a trader's belief can be represented by a point \( \gamma \) which identifies a particular conjugate distribution \( H(\theta | \gamma) \) over the set of possible payoff distributions \( F(X; \theta) \). The point \( \gamma \) lies in some vector space, possibly infinite dimensional. To allow for the pos-

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\(^8\)This assumption rules out more complex intertemporal strategies such as buying now at a price above expected value in hopes of selling later (to a 'greater fool') at an even higher price. We are happy to rule out such strategies, which underlie the main 'rational bubbles' models as well as many irrational bubbles stories, in the interest of simplicity and (more importantly) to emphasize that imperfect information aggregation by itself can lead to bubbles.
sibility of complete ignorance, we postulate a belief $\gamma^0$ such that $H(\theta | \gamma^0)$ is diffuse (DeGroot (1970), p. 190). The previous section used the example $\gamma = (a, b)$ in the conjugate Gamma distribution for the exponential parameter. In this case, complete ignorance is represented by $\gamma^0 = \lim_{b \to 0}(2, b)$, which says that the exponential parameter $\lambda$ has the (improper) uniform distribution on $[0, \infty)$. We summarize this discussion in the following general assumption.

(B0) There is some family of distributions $H(\theta | \gamma)$, $\gamma$ in some convex open subset $\Gamma$ of a topological vector space, such that at any time $t$ any trader $i$'s beliefs regarding the venture's payoff can be summarized by $H(\theta | \gamma_i)$ for some $\gamma_i \in \Gamma$. There is some element $\gamma^0 \in \Gamma$ such that $H(\theta | \gamma^0)$ is diffuse.

Traders' beliefs change over time largely in response to private news. We assume for simplicity that each trader $i$ receives a non-trivial private news message $z_i$ in period $t$ with probability $\pi > 0$, and no message (sometimes conventionally denoted $z_i = 0$) with probability $1 - \pi$. We have in mind the view news could include such diverse events as 'a reliable friend says the venture's director is incompetent' or 'Newsweek magazine runs a favorable article on a similar venture'. We let $S$ denote the set of possible news messages. For many purposes we require no further structure on $S$, but sometimes we assume that $z_i$ is independently drawn from a specific distribution $G(z | \theta, \bar{a})$ on $S$, where $\theta$ is the true but unknown value of $\theta$ in the payoff distribution and $\bar{a}$ is the precision (i.e., $1$/variance) of $G$.

Traders' beliefs may differ because of different prior beliefs, or, more importantly, because of different private information. In the interest of parsimony we assume that beliefs do not diverge because of idiosyncratic information processing. That is, we assume that all traders use the same updating function $\psi$ when responding to private news $z$ and to (publicly observed) price changes $\Delta p$. Formally,

(B1) Each trader updates his beliefs by means of the same continuous updating function $\psi: \Gamma \times S \times R \to \Gamma$, so $\gamma_{i+1} = \psi(\gamma_i, z_i, \Delta p_{i+1})$.

In the previous section we used an updating function which can now be expressed as $\psi((a, b), z, y) = (a + \bar{a}, b + \bar{a}z + cy)$. Simple formulae of this type can be found for special conjugate families of distributions $G$ and $H$ (DeGroot (1970), Ch. 9), but in general $\psi$ has no closed form and $\Gamma$ is infinite dimensional. In rational expectations equilibrium (REE) models $\psi$, typically is shown to exist by a fixed point argument. In our model we certainly do not exclude the possibility that traders use REE updating procedures, but we prefer not to impose the very strong assumptions underlying REE: that traders fully understand the market environment, have unlimited computational powers, know (as common knowledge) that the same is true for other traders, etc. In light of recent empirical work
(e.g., Dwyer et al. (1990)) we feel more comfortable in assuming that traders' expectations are unbiased and consistent, but not necessarily minimum variance.

To formalize this view we need some further notation. Let \( q^* \) denote the extension of \( q \) to updating over several periods, defined inductively by \( q^*(y; z_1, z_m; y_1, \ldots, y_n) = q(q^*(y; z_1, \ldots, z_{m+1}; y_1, \ldots, y_{n+1})) \). Slightly abusing notation, we abbreviate \( q^*(y; z_1, \ldots, z_m; 0) \) as \( q^*(z_1, \ldots, z_m) \) and abbreviate \( q^*(y; 0; y_1, \ldots, y_n) \) as \( q^*(y_1, \ldots, y_n) \) when \( y \) is understood.

Let \( \theta \) denote the unknown index value for the true payoff distribution, so with perfect (but unobtainable) information the payoff expectation would be \( \bar{X} = E(X; \theta) = \int x F(dx; \theta) \). We denote trader \( i \)'s current payoff expectation by \( \bar{X}_i = E(X|\gamma_i) = \int x F(dx; \theta) H(d\theta; \gamma_i) \). The expectation function \( \phi \) used by traders is induced by the belief updating function \( \psi \), so \( \phi(\gamma, z, y) = E(X|\psi(\gamma, z, y)) \), and we denote its multi-period extension by \( \phi^* \). In particular, the 'sample expectation' for the observation \( z \) is denoted by \( \phi^*(z) = \phi^*(\gamma^0, z, 0) \); recall that \( \gamma^0 \) indicates a diffuse prior.

Our main rationality assumption now can be formalized as follows:

(B2) News messages are interpreted so as to provide consistent and unbiased information regarding the true expected payoff \( \bar{X} = E(X; \theta) \). That is:

a. \( \bar{X} = E\phi^*(z) \) (so news interpretation is unbiased);

b. For each \( \gamma \in \Gamma \), the posterior expectation \( \phi(\gamma, z, 0) \) is strictly increasing in the sample expectation \( \phi^*(z) \), and lies between \( \phi^*(z) \) and the prior expectation \( \phi(\gamma, 0, 0) \) (so news is informative); and

c. For any fixed \( \gamma \in \Gamma \), the expectation \( \phi^*(z_1, \ldots, z_m) \rightarrow \bar{X} \) with probability 1 as \( n \rightarrow \infty \) (so news is consistent).

Now consider how public information (\( \Delta p \)) affects expectations. In the example of Section 2 we assume that the expectation function is \( \phi(a_0, b_0, z, y) = (b + cy)/(a - 1) \), where \( a = a_0 + \tilde{a} \) and \( b = b_0 + n\tilde{a} \). In this case the sensitivity of expectations to public information is \( \partial \phi/\partial y = \phi_3 = c/(a - 1) > 0 \). Note that \( \phi_3 < 1 \) as long as \( c < a - 1 = a_0 + n\tilde{a} \); that is, as long as the number \( n \) of messages received is sufficiently large, given the initial \((t = 0)\) precision \( a_0 \) and the news precision \( \tilde{a} \). For the general model we allow complicated, possibly non-linear and time dependent, responses of expectations to price changes. We assume only that the responses are non-negative and not overly sensitive. Information aggregation can be short-circuited if traders allow their own private information to be outweighed by public information (Bikchandani, Hirshleifer and Welch (1991)). To rule out 'cascade' or 'lemming' behavior we assume \( \phi_3 \) is strictly less than 1. (When \( \phi_3 \) is greater than or equal to 1, we can have unstable situations in which a price rise of 1 cent causes expectations to rise by, say, 2 cents, causing a 2 cent price rise, in turn causing expectations to rise by 4 cents, \ldots. ) Our formal assumption, then, is
The expectation function $\phi$ is continuously differentiable and, for some $\delta > 0$, satisfies $0 \geq \phi_y \geq 1 - \delta$ for all $y$, $z$, and $\gamma$.

Assumptions (B0)-(B3) involve some mild technical conditions. (The differentiability assumptions probably are the strongest, and they could be replaced by less restrictive but more cumbersome Lipshitz continuity conditions.) For some purposes we use other mild technical conditions regarding boundedness and onto-ness. Specifically,

1. **(B4)** a. There is some $B > 0$ such that $E(X|\gamma) < B$ for all $\gamma \in \Gamma$. (So payoff expectations are uniformly bounded.)

   b. For each $\gamma \in \Gamma$ there is some $z \in S$ such that $E(X|\gamma) = \phi^*(z)$. (So all parameter values $\gamma$ give expectations that conceivably could be news-justified.)

Now that all elements of the model are in place we can define the fundamental as the sample expectation based on all news previously received by all traders, i.e., as $f_+ = \phi^*(z_1, \ldots, z_M)$. Then the bubble at time $t$ is the difference between the current price and fundamental, $b_t = p_t - f_t$. We begin our analysis with a derivation of the optimal bid, taking into account the 'winner's curse' problem discussed in the previous section.

**Proposition 1.** Under Assumptions B0–B3, the equation $y = \phi(y, 0, y - p_t)$ has a unique solution $y^*$ for each $p_t > 0$ and $\gamma_0 \in \Gamma$. If $p_t$ is the most recently announced price and $\gamma_0$ describes trader i's current beliefs regarding the asset payoff, then his optimal bid is $v_{it} = y^*$.

The proof of this and the other propositions appear in the Appendix. The main idea can be seen in Figure 4: to avoid the winner’s curse, the optimal bid $y^*$ differs from the current expectation $\bar{x}_{it}$ to the extent that the current expectation differs from current price.

Before turning to price determination, it may be worth examining in more detail how the optimal bid depends on the updating process $\psi$. In the context of i.i.d. private news $z_{it}$, it is natural to assume that $\psi^*(z_1, \ldots, z_M)$ is a symmetric function, i.e., the order in which these messages arrive is irrelevant. A corresponding (but more problematic) symmetric property for the public information embodied in price changes is price path invariance (PPI). We say $\psi$ is (or has) PPI if $\psi^*(\gamma; z_1, \ldots, z_n; y_0, \ldots, y_t)$ depends on the price changes $(y_0, \ldots, y_t)$ only through the sum $\Sigma_{t=0}^T y_t$. That is, the order in which price changes occur is irrelevant, and only their net effect matters. Thus the current price $p_t$ (together with the time 0 price $p_0$) is a sufficient statistic for the sequence of price changes under PPI since $p_t - p_0 = \sum \Delta p_r$, as is the case in many REE models.

An important alternative to PPI is that traders may respond less strongly to prospective than to actual price changes. There is considerable evidence for such behavior in related economic contexts e.g. Arrow (1981), Cox, Smith and Walker (1983), and Grether (1978); specifically we are suggest-
ing that traders may not fully adjust for the winner’s curse, as documented by Kagel and Levin (1986). To formalize this idea, let $\bar{\phi}$ be the expectation function obtained from $\phi$ by replacing its last argument $\Delta p_{t+1}$ by a hypothetical price change which has not (yet) been observed. If $0 \geq \phi_3 < \phi_3$ and PPI holds for observed price changes, then we say that traders discount hypothetical price changes, or DHP holds.\textsuperscript{9} In Figure 4, this

\textsuperscript{9} Unlike the ‘positive feedback traders’ of DeLong et al. (1990), our DHP traders do not blindly extrapolate price trends. They merely weight the evidence regarding the expected final payoff contained in $\Delta p$ against other evidence from history and own private news. Indeed, our DHP traders may well understate the inferences which can be drawn from $\Delta p$.  

\textbf{Fig. 4. The Optimal Bid Function.}
amounts to rotating the curve $\phi$ clockwise around the point $(p, \hat{x})$, not allowing it past the horizontal.

**Proposition 2.** Assume B0–B3 hold. Under PPI, a trader's optimal bid $v_\alpha$ depends only on private news $\{z_{\alpha1}, \ldots, z_{\alpha}\}$ and is independent of observed prices $\{p_1, \ldots, p\}$. Under DHP the optimal bid $v_\alpha$ is an increasing function of the most recent price change $\Delta p$, ceteris paribus.

In the DHP case, we note that $\phi_\alpha > 0$ so that (absent contrary news) $\Delta \hat{x}_\alpha \leq 0$ and $\Delta v_\alpha \leq 0$ as $\Delta p \leq 0$, i.e., expectations and bids respond in the same direction to price changes. This property seems quite natural to us, and we were surprised to discover that it does not hold under PPI. Indeed Proposition 2 shows that under PPI optimal bids will not respond at all to price changes, even when expectations respond quite strongly. The main idea is that $y^* - p_\alpha$ is a sufficient statistic under PPI for all price changes in the equation that defines $v_\alpha$, but $y^*$ is independent of observed prices. In retrospect, this suggests the intuition that under PPI traders care only about the prospective transaction price, which does not really depend on previous price changes.

We are now prepared to characterize the prices that emerge from our Clearinghouse market, in terms of traders’ prior and posterior expectations, $\hat{x}_{\alpha t+1} = \phi(y_\alpha t, 0, \Delta p_{\alpha t+1})$ and $\hat{x}_{\alpha t+1} = \phi(y_\alpha t, z_{\alpha t}, \Delta p_{\alpha t+1}) = \phi(y_\alpha t+1, 0, 0)$. We let $H$ denote the set of shareholders, $\bar{H}$ denote the set of non-shareholders, and $\# S$ denote the number of elements in the set $S$.

**Proposition 3.** Under the rules of the simple Clearinghouse market, we have $p_{\alpha t+1} = v_{(M)\alpha t}$ for $t = 0, 1, \ldots, T-2$. Given optimal bidding under B0–B3, shareholders' posterior expectations are no lower than those of nonshareholders, i.e., $H_{\alpha t+1} \subset \{i: i_{\alpha t+1} \geq p_{\alpha t+1}\}$ and $\bar{H}_{\alpha t+1} \subset \{i: i_{\alpha t+1} \leq p_{\alpha t+1}\}$. Under strict PPI for $\psi$ we have $p_{\alpha t+1} = \hat{x}_{(M)\alpha t+1}$, while $p_{\alpha t+1}$ is between $p_\alpha$ and $X^{(M)\alpha t+1}$ under DHP.

Proposition 3 allows us to investigate price behavior. Recall $\hat{X} = E(X | \hat{\theta})$ is the 'true value' of the asset. Our next result tells us that the fundamental $f_\alpha$ and price $p_\alpha$ are both consistent estimators of the asset’s true value $\hat{X}$, and are essentially unbiased (i.e., any bias is inherited from biased initial priors). Therefore, bubbles are zero on average and have a variance that becomes arbitrarily small when sufficient information accumulates.

**Proposition 4.** Under B0–B4, both $p_\alpha$ and $f_\alpha$ converge to $\hat{X}$ with probability 1 as $t \to \infty$. If the prior distributions for $\theta$ are unbiased, i.e., if $E(X | \gamma_{(1)}) = \hat{X}$ for $i = 1, \ldots, N_i$ then the unconditional expectations of $p_\alpha$ and $f_\alpha$ are both $\hat{X}$ for all $t = 0, 1, 2, \ldots$.

In view of this proposition, bubbles are significant only to the extent that $\text{Var} p_\alpha$ far exceeds $\text{Var} f_\alpha$, i.e., $p_\alpha$ is much less efficient than $f_\alpha$. It turns out that this relative efficiency depends on the nature of the private news arrival process, which may be characterized largely by the precision $\tilde{a} = 1/$
Var \( z \) of a news message and the probability \( \pi \) of its arrival during a trading day. The \textit{intensity} of the process, \( I = \tilde{\alpha} \pi \), is the average rate at which a trader becomes (privately) informed. One might be tempted to consider the comparative statics of news intensity, but a moment's reflection reveals that greater intensity is equivalent merely to more frequent market clearings — one just redefines the time scale to obtain unit intensity. Therefore the interesting comparative statics concern lumpiness, the extent to which news arrival is rare and decisive versus frequent but ambiguous. We say that a news process \((\pi, a)\) is \textit{lumpier} than another \((\tilde{\pi}, \tilde{a})\) with the same intensity if \( \pi < \tilde{\pi} \) (so \( a > \tilde{a} \)). The limiting cases are \( \pi = 1 \), in which case traders are always equally well informed, and \( \pi \to 0 \), in which case a few traders may be far better informed than most.

One can obtain the following approximation (see the Appendix) for the excess variance ratio:

\[
EV_i \equiv \frac{\text{Var} \ p_i}{\text{Var} \ f_i} \approx k \frac{1-(1-\pi)^{i+1}}{1-(1-\pi)^{N+1}},
\]

where \( k \) depends on \( \phi_3 \), e.g. \( k \approx 1.57 \) for \( \phi_3 \approx 0 \). Since the RHS is decreasing in \( \pi \) we conclude that lumpier news tends to decrease the efficiency of \( p_i \) relative to \( f_i \) as an estimator of \( \bar{X} \), so bubbles become increasingly important. Indeed, by L'Hospital's rule, the RHS \( \to \infty \) as \( \pi \to 0 \), so bubbles tend to dominate price movements in the case of extremely lumpy news (i.e., extreme asymmetries in traders' private information). This analytic result confirms the simulation evidence presented in Table 1. The intuition is that \( p_i \) is essentially a \textit{median} of traders expectations of \( \bar{X} \) while \( f_i \) is an appropriately weighted \textit{mean}, which is much more efficient when the (information-received) weights are very unequal.

Recall that some simulations suggested that bubbles may be self-feeding and/or have some built-in tendency to burst. We now proceed to consider these matters more formally.

\textit{Proposition 5.} Assume PPI holds. Then \( E(\Delta p_{t+1} | p_t, \tilde{\theta}) \) is a decreasing function of \( p_t \) whose sign agrees with that of \( \bar{X} - p_t \).

The essential idea here is that, to the extent that \( p_t \) is above the asset's true value \( \bar{X} \), unbiased private news is more likely to lower expectations and bids of shareholders and less likely to raise expectations and bids of nonshareholders, thus increasing the probability of negative price changes. Similarly, positive (and larger) price changes are more likely to the extent that \( p_t \) is below \( \bar{X} \). Thus we have mean-reversion tendencies.

The same tendencies are also at work under DHP, and actually are reinforced by another mechanism. In view of the relative efficiency of \( f_i \) (and Lemma 5.1 in the Appendix) one has a positive correlation of bubbles with \((p_i - \bar{X})\). To the extent that \( b_i > 0 \), nonshareholders tend to be 'better informed' and less responsive to news than shareholders. Inasmuch
as price decreases arise from lower bids by shareholders and the absence of higher bids by nonshareholders, this differential in news-responsiveness (which requires \( \pi < 1 \)) will tend to lead to price decreases. Similarly, price increases are more likely when \( b_r < 0 \). In the interest of brevity we do not attempt to formalize this point.

Perhaps the most interesting question is when bubbles are \textit{self-feeding}, which we define as the property that \( E(\Delta b_{t+1} | \Delta p_t) \) is an increasing function of \( \Delta p_t \).

\textit{Proposition 6.} Suppose news arrival is not certain \((\pi < 1)\) and DHP holds. Then bubbles are self-feeding when the asset’s fundamental value \( f_t \) is sufficiently near its true value \( \hat{X} \).

The mechanism driving this result may be worth elaborating. Figure 5 shows the most likely events (called Events \( \pm 1 \)) which can produce a transaction: a near-median trader responds more or less strongly to public news than the median trader, and hence transacts with him. Given our assumption of a common updating function \( \psi \), it must be the case that our less responsive trader was better informed (i.e., had a larger number of news messages). Hence one could say that self-feeding bubbles arise in our model as less informed expectations (or bids) overtake better informed expectations (bids).

Clearly this mechanism operates more strongly when \( \pi \) is small, so both self-feeding bubbles as well as excess variance are more important when news is lumpy. On the other hand, PPI makes \( \Delta y_{i+1} \) independent of \( \Delta p_t \) by Proposition 2, so evidently bubbles are not self-feeding in this case.

\textbf{Event -1}

\textbf{Event +1}

Fig. 5. Price Transition Events.
Our main argument may be summarized as follows. If asset price is not a sufficient statistic for aggregate information, traders nonetheless will generally find it informative, and find it in their interest to respond to it. Under these circumstances, prices will not a priori be biased away from the fundamental value (which is based on the aggregate of all current information), but will generally exhibit higher variance. That is, asset prices will be more volatile than the strong efficient markets hypothesis (EMH) would suggest. Furthermore, bubbles (i.e., discrepancies between price and fundamental) may be self-feeding, in the sense that an increase (or decrease) tends to provoke a further increase (or decrease). Bubbles of the sort we examine are eventually self-limiting in that the probability of reversal increases with magnitude. The last two features imply overshooting: when a positive (or negative) bubble disappears, its momentum tends to produce a negative (or positive) bubble. These market inefficiencies are not arbitragable by market participants, who have access only to observed prices and own private information, not aggregate information.

We developed this argument in terms of a theoretical asset market model, together with a parametric illustration and computer simulations, featuring news (dispersed private information arrival) and Bayesian traders oriented to long-term asset value. The model suggests that prices will be excessively volatile (relative to fundamental values) to the extent that (1) news is of high quality relative to prior information, so it induces large revisions in asset value estimates; (2) news arrives infrequently, so individual asset value estimates remain uncertain over much of the life of the asset, and (3) there is not so much background noise (e.g. liquidity shocks) that price changes are regarded as uninformative. In our model, self-feeding (and overshooting) arise if also (4) traders underweight prospective price changes relative to actual price changes. Thus the empirical implications concern the relation between news arrival (and news processing) and certain types of price behavior.

Although difficult to identify by econometric analysis of historical data, the existence of excess price volatility, self-feeding and overshooting have been suggested by many participants and analysts of contemporary asset markets. For example, with respect to excess price volatility, the ‘variance bounds’ literature attempted to demonstrate that US stock market prices had higher variance than could be accounted for by rationally valued underlying dividend or earnings streams — see Shiller (1981) or LeRoy

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10 Their variance bound is based on the fact that conditioning on a larger information set can only increase the conditional variance. This fact has no bite in our model because our asset price is the expectation of a particular trader (the median) whose identity changes over time, and our traders have heterogeneous, non-nested information sets.
and Porter (1981), for instance. The present consensus appears to be that these empirical tests are inconclusive because the theoretical models they employ rely too heavily on the exogeneity and stationarity of the relevant time series (e.g. dividend or earnings streams) — see Marsh and Merton (1986) or Kleidon (1984) for instance. A more direct test requires data on the information employed by investors, but of course such data are not normally available to the econometrician. Roll (1984) studied perhaps the most accessible case: he argued persuasively that, over the period he considers, the only information relevant to short-run fluctuations in orange juice futures prices is the predicted temperature in central Florida, yet 'surprises' in the latter variable explain only a small fraction of the observed daily price variability. (He also points out that price variance across weekends should be three times as large as across weekdays under the EMM, but it is actually only about 1.5 times as large.)

There is also a set of recent articles that seek empirical tests of the exponential rational bubbles literature. For example, Meese (1986), employing a specification test suggested by West (1984), finds monthly foreign exchange rate data reject a joint hypothesis of no bubbles and a stable driving process for a monetary exchange rate model. He carefully qualifies his tentative conclusion that bubbles are present. On the other hand, Hamilton and Whiteman (1986) argue strongly that bubbles are impossible to detect econometrically when market participants may respond to variables not observed by the econometrician, and that previous empirical detection of bubbles (and excess volatility) was invalid.

A different empirical approach involves laboratory experiments with asset markets in which trader's information is controlled and therefore completely observable. Smith, Suchanek and Williams (1988) report massive positive (but non-exponential) bubbles in a double auction asset market even when the news arrival process is trivial. It remains to be seen whether such phenomena persist under conditions more closely resembling those of our model, but we regard the experimental approach as the most promising empirical technique for studying information aggregation and asset price dynamics.

Our model as presented in this paper involves a single payoff, extreme indivisibility, and no exogenous public information, and therefore can not be applied directly to most contemporary asset markets. However, intuition and some preliminary analysis suggest that its main conclusions survive considerable generalization. For example, most important securities have a payoff stream which extends over time. The cost of modeling such securities is more complex notation and calculations. Preliminary work with models of this sort reveals no major new insights, but does point up the convenient fact that the expected present value of an infinite-lived payoff stream need not have a variance that decreases over time, so excess volatility can be measured directly rather than through ratios as in Table 1. Relaxing the indivisibility and risk neutrality assump-
tions blurs the distinction between the median and mean, but some preliminary work indicates that (except under some special parametric specifications) price is still not a sufficient statistic for all private information and our main argument remains valid. Exogenous public information is not usually transparent; to the extent that asset market participants differ in their evaluations of public news it can be regarded as independent sample information. (In Bayesian terms, participants may have different priors as well as different likelihood functions for the asset value implications of news, so even weather forecasts may present an information aggregation problem for the orange juice futures market.) The impact of transparent exogenous public information perhaps can be modeled in much the same way as the impact of a price change, but this also remains a matter for further investigation.

We are not the first to argue that asset prices do not fully aggregate information available to market participants; see Grossman (1976) and especially Figlewski (1982) for an excess variance argument. As we see it, our main contribution is to show how excess variance due to imperfect information aggregation relates to asset price bubbles that are self-feeding, overshoot, etc. and to show how these phenomena arise from the underlying information conditions and market structure. Once bubbles of our sort become prevalent, some traders may find it worthwhile to pursue short-run technical strategies, in which case ‘beauty contest’ bubbles become plausible.\(^\text{11}\) Thus our theory of bubbles does not replace the older theories surveyed in the introduction, but rather suggests conditions under which these theories may become germane.

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Received July 1990, revision accepted October 1991

APPENDIX

Lemma 1. Let \(a, b, c, d\) be positive numbers. Then
\[
\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}.
\]

Proof: \[
\frac{a+c}{b+d} = \left(\frac{b}{b+d}\right) \frac{a}{b} + \left(\frac{d}{b+d}\right) \frac{c}{d} = \alpha \frac{a}{b} + (1-\alpha) \frac{c}{d}
\]
for \(0 < \alpha = \frac{b}{b+d} < 1\), and the conclusions follows immediately.

\(^{11}\) It is precisely this insight which DeLong et al. (1990) exploit.
Proof of Proposition 1: Since $X$ is non-negative, $E(X \mid \psi(y, 0, y - p)) = \phi(y, 0, y - p) \geq 0$ for $y = 0$ and all $p \geq 0$ and $y \in \Gamma$. By (B3) we have $\phi(y, 0, y - p) < y$ for $y$ sufficiently large. Hence (employing B1 and B3) the function $h(y) = \phi(y, 0, y - p) - y$ is continuous, decreasing, positive at $y = 0$, and negative for $y$ sufficiently large. Therefore by the intermediate value theorem, $h(y)$ has a unique root $y^*$, as required. See Figure 2, and note that $h(y) \geq 0$ if $y \leq y^*$.

The optimization problem for a non-shareholder is

$$\max_{v \geq 0} E[(X - p_{t+1}) I[p_{t+1} \leq v]) \mid y, [p_{t+1} \leq v]],$$

where $I[e] = 1$ if the event $e$ occurs and is 0 otherwise. Thus the maximand is

$$\int_{0}^{y} [E(X \mid \psi(y, 0, y - p)) - y] dF^*(y) = \int_{0}^{y} h(p_{t+1}) dF^*(p_{t+1}),$$

where $F^*$ is the (subjective) distribution of $p_{t+1}$. Clearly, whatever $F^*$ might be, the maximum is achieved by integrating over $\{y : h(y) \geq 0\}$, so $v = y^*$ is optimal.

The optimization problem for a shareholder $i$ is

$$\max_{v \geq 0} E[(p_{t+1} - X) I[p_{t+1} \geq v]) \mid y, [p_{t+1} \geq v]],$$

a very similar analysis also leads to the same conclusion in this case. QED

Proof of Proposition 2. For arbitrary $y$ (suppressed) we know by (B3) that the following function is differentiable:

$$h_i(y, \rho) = \phi^*(\Delta p_1, \ldots, \Delta p_{t-1}, \rho, y - (p_{t-1} + \rho)) - y,$$

where $\rho = \Delta p_i$ is the most recently observed price change. By Proposition 1, the optimal bid $v_i$ is $y^* = h_i^{-1}(0, \rho)$, the inverse function taken with respect to the first (y) argument of $h$. Under PPI, $h_i$ is independent of $\rho$, so $y^*$ is also independent of $\rho = \Delta p_i$. By induction, it is also independent of $\Delta p_t$, $t = 1, \ldots, t-1$ as well, so the first part of the proposition follows. Under DHP we see that $h_i$ is increasing in $\rho$ since the last argument $y - (p_{t+1} + \rho)$ of $\phi^*$ is hypothetical but the previous argument $\rho$ is not. Consequently $y^*$ is also increasing in $\rho = \Delta p_{t+1}$. QED

Proof of Proposition 3. Assume to begin with that no two $v_i$'s coincide. Recall that traders are assumed not to attempt to exercise monopoly power, and that $S_i(p) = \# \{i \in H_i : v_i \leq p\}$ and $D_i(p) = \# \{i \in H_i : v_i > p\}$ are the supply and demand functions. For $p > v_{i(t)}$, one clearly has $S_i(p) = M$ and $D_i(p) = 0$. As $p$ decreases from $p^+ = v_{i(t)} + \epsilon$ to $p^- = v_{i(t)} - \epsilon$, we have either $S_i(p^-) = S_i(p^+) - 1$ (when $i \in H_i$) or $D_i(p^-) = D_i(p^+) + 1$ (when
i ∈ \tilde{H}_t). Consequently \( S(p) > D(p) \) iff \( p > v_{(M)_t} \) — see Figure 1. It then follows from the definitions that \( p_{t+1} = v_{(M)_t} \) and \( H_{t+1} = \{ i : v_{u_i} \geq p_{t+1} \} \) while \( \tilde{H}_{t+1} = \{ i : v_{\tilde{u}_i} < p_{t+1} \} \). Now \( \tilde{x}_{u_i} = \phi(\gamma_{u_i}, 0, \Delta p_{t+1}) > v_{u_i} \) iff \( p_{t+1} > y^* = v_{u_i} \) by (B3) — see Figure 2 — so one can replace \( v_{u_i} \) by \( \tilde{x}_{u_i} \) in the previous sentence. By the same argument, \( v_{u_i} = p_{t+1} \) implies \( \tilde{x}_{u_i} = v_{u_i} \), so we conclude that \( p_{t+1} = \tilde{x}_{(M)_t+1} \) under PPI. The weaker statements in the proposition then follow from assuming only DHP and allowing for the possibility of two or more traders bidding \( v_{(M)_t} \).

Proof of Proposition 4. Let \( n_{u_i} = \# \) messages received by trader \( i \) up to time \( t \), and let \( n_t = \sum_{i=1}^{n_{u_i}} n_{u_i} \). Since \( n_t \to \infty \) with probability 1 as \( t \to \infty \), it follows from (B2) and the definition of \( f_i \) that \( f_i \) is unbiased and consistent. Suppose now that \( \phi_3 = 0 \), i.e., prices are uninformative. Since by (B4) an unbiased prior \( \gamma_{(0)} \) can be regarded as \( \gamma(z_{(0)}) \) for some realization \( z_{(0)} \) the news process and since \( n_{u_i} \to \infty \), the same argument shows that \( \tilde{x}_{u_i} \) is also unbiased and consistent for each \( i \), so by Proposition 3, we obtain the same conclusion for \( p_{u_i} \). When \( \phi_3 > 0 \), the argument is more delicate; essentially we must show that the news \( \{ z_{i1}, ..., z_{in} \} \) eventually dominates the effect of price changes. We first note that \( p_i \leq \max[\tilde{x}_{u_i} : i = 1, ..., n] \leq \sup[\mathbb{E}(X|\gamma) : \gamma \in \Gamma] \leq B \), the inequalities from Proposition 3, the definition of \( \tilde{x} \), and (B4) respectively. Consequently \( \Sigma_{t \leq i} \Delta p_{t_i} = p_{t_i} - p_0 \leq B \), so the bound \( \phi_3 \leq 1 - \delta \) from (B3) then yields a uniform bound on \( \mathbb{E}(X|\gamma(z_{(0)}) \). Now (B4) implies that \( \mathbb{E}(X|\gamma(z_{(0)}) \) exceeds this bound with positive probability, but (B2) assures us that even in this event we have \( \mathbb{E}(X|\gamma(z_{(0)}, z_{1}, ..., z_{n}) \to \mathbb{E}(X|\tilde{\theta}) \) as \( t \to \infty \) with probability 1, for \( (z_{i1}, ..., z_{in}) \) the actual news received by a trader \( i \). Hence by Propositions 1 and 3 and the Dominated Convergence Theorem we must have \( \mathbb{E}(X|\gamma_{u_i}) = \mathbb{E}(X|\tilde{\theta}) \) for each \( i \), establishing the consistency of each \( \tilde{x}_{u_i} \) and hence \( p_{u_i} \). Finally, for unbiasedness, note that \( \tilde{x}_{(0)} \) are unbiased by the hypothesis on \( \gamma_{(0)} \). Under the inductive assumption that \( \mathbb{E}p_{t_i} = \mathbb{E}X_i = \mathbb{E}(X|\tilde{\theta}) \) for all \( i \), we note that \( \mathbb{E}x_{u_{i}+1} \) is a convex combination of the latter variable and \( \mathbb{E}(X|\gamma(z_{u_{i}+1})) = \mathbb{E}(X|\tilde{\theta}) \), since \( \mathbb{E}(\Delta p_{t_i}) = 0 \), so \( \mathbb{E}x_{u_{i}+1} = \mathbb{E}(X|\tilde{\theta}) \) also. Consequently \( \mathbb{E}p_{t_i+1} = \mathbb{E}(X|\tilde{\theta}) \). QED

We now analyze the dependence of the bubble index \( EV_t = \text{Var} p_t / \text{Var} f_t \) on the parameters describing the news process. Assume that news \( z_{u_i} \) is iid \( G(z|a, \theta) \), where \( a = \text{Var}^{-1} z \) is the precision of a news message, and \( \pi \) is its probability of arrival. Thus the intensity of news, defined as \( I = \tilde{a} \pi \), summarizes the rate at which an individual becomes informed. By redefining the time scale \( t \), we can (apart from the effect of more frequent market clearings) normalize on a given news intensity \( I_0 \). Recall that a news process \( (\pi, a) \) is lumper than a process \( (\tilde{\pi}, \tilde{a}) \) with the same intensity \( (I_0 = a \pi = \tilde{a} \tilde{\pi}) \) if \( \pi < \tilde{\pi} \). The limiting cases are \( \pi = 1 \) in which case all traders are always equally well informed, and \( \pi = 0 \) in which case news is very lumpy and typically a few traders are far better informed than others.
It is convenient in this analysis to assume that prior beliefs at $t = 0$ arise from a preliminary message $z_{0i} \sim \text{iid } G(z | a, \theta)$. Now traders' expectations at time $t$ are described by $S' = \{x_i : i = 1, \ldots, N\}$; for the case $\phi_3 = 0$ (i.e., prices regarded as uninformative), $S_i$ is an iid random sample from a distribution $G_i^*$ derived from $G$ and the binomial distribution with parameters $\pi$ and $t$, as noted below. We employ the decomposition $EV_i = (\text{Var } P_i / VS_i)(VS_i / \text{Var } f_i)$, where $VS_i$ is the variance of $S' - \text{sample mean}$, and analyze the two factors separately.

The case $\phi_3 = 0$ can be analyzed rather explicitly. Let $n_{ii}$ be the number of actual (not preliminary) news messages received by trader $i$ by the end of period $t$; clearly it has the binomial distribution so

$$\Pr[n_{ii} = n] = \binom{n}{n} \pi^n (1 - \pi)^{t-n}.$$  

The conditional variance of $\tilde{x}_{ii}$ is $\text{Var}(\tilde{x}_{ii} | n_{ii}) = a^{-1}(n_{ii} + 1)^{-1}$; it follows that the unconditional variance is $\text{Var} \tilde{x}_{ii} = a^{-1}E(1/(n_{ii} + 1) | \pi, t)$. But

$$E\left(\frac{1}{n+1} \middle| \pi, t \right) = \frac{1}{n+1} \text{ is the sample mean, and analyze the two factors separately.}$$  

Consequently,

$$VS_i = \frac{1}{N} \text{Var} \tilde{x}_{ii} = \frac{1}{Na} E\left(\frac{1}{n+1} \middle| \pi, t \right) = \frac{1 - (1 - \pi)^{t+1}}{(t+1)\pi}.$$  

Now $\text{Var } f_i$ can be computed in an analogous fashion. For $n_i = \sum n_{ii}$, the total number of messages incorporated in $f_i$ (counting, as we should in this context, the preliminary messages) is $n_i + N$, so the conditional variance of $f_i$ is $a^{-1}(n_i + N)^{-1}$ and the unconditional variance is

$$\text{Var } f_i = \frac{1}{a} E\left(\frac{1}{n_i + N} \middle| \pi, Nt \right) = \frac{1}{a} \frac{\pi t + 1}{N} E\left(\frac{1}{n_i + 1} \middle| \pi, Nt \right)$$

$$= \frac{(\pi t + 1/N)(1 - (1 - \pi)^{Nt+1})}{(\pi t + 1)(Nt + 1)\pi a}.$$  

We employed the approximation

$$\frac{1}{n_i + N} \approx \frac{1}{n_i + 1} \frac{n_i/N + 1/N}{n_i + 1} \approx \frac{1}{n_i + 1} \frac{\pi t + 1}{\pi t + 1},$$
valid for $t$ sufficiently large that $\pi = n_t/N_t$. It follows that the second factor in the decomposition of $EV_i$ is

$$
VS_i/\text{Var } f_i \approx \frac{(Nt+1)}{N(t+1)} \frac{(nt+1)}{N(t+1/N)} \frac{(1-(1-\pi)^{t+1})}{(1-(1-\pi)^{Nt+1})} \approx \frac{1-(1-\pi)^{t+1}}{1-(1-\pi)^{Nt+1}}.
$$

The last formula clearly shows that the second factor is decreasing in $\pi$ (i.e., increasing in the lumpiness of news), is equal to 1 for $\pi = 1$, and (by L'Hospital's rule) approaches $\infty$ as $\pi \to 0$. These conclusions can also be confirmed more laboriously from the exact formulae. The result may be familiar from statistical theory: $VS_i$ corresponds to an OLS estimator and $\text{Var } f_i$ corresponds to a GLS estimator employing the known heteroskedasticity of the 'sample' $S$.

As for the first factor $\text{Var } p_i/VS_i$, Proposition 2 implies that we are comparing the efficiency of the sample median (or more generally, the $M^{th}$ order statistic) to that of the sample mean. It is known that this ratio depends on the shape of the underlying distribution $G^*$ but is rather insensitive to its location or scale, and therefore is, to a first approximation, independent of $t$, $\pi$ and $\alpha$. Specifically, Fisz (1963, p. 383) shows that if $N$ is reasonably large, then $\text{Var } p_i$ is approximately $\langle M/N \rangle (1-M/N) N^{-1} g_i(\bar{x})^2$, where $g_i$ is the density of $G^*$, assumed continuous at the $M/N$ - quantile $\bar{x}$. By (B2), $\bar{x}$ is the mean of $G^*$, and the Central Limit Theorem suggests that $g_i(\bar{x}) \approx (2\pi \sigma_i^2)^{-1/2}$, where here $\pi \approx 3.14$ and $\sigma_i^2$ is the variance of $G^*$. Hence for the case $M = N/2$ we have

$$
\text{Var } p_i \approx \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2N} \cdot 2\pi \sigma_i^2 = \frac{\pi \sigma_i^2}{2N}.
$$

Since in the present case $VS_i = \sigma_i^2/N$, we obtain $Vp_i/VS_i \approx \pi/2 \approx 1.57$ as an approximation valid for reasonably large $N$ and $\pi t$.

We do not attempt an explicit analysis of the case $\phi_3 > 0$ (informative news), but note that $\phi_3$ should depend on news intensity rather than news lumpiness. Indeed, $\phi$ is presumably chosen to minimize $\text{Var } x_{ui}$; to the extent this is successful (and 'noise' does not overwhelm 'signal'), $VS_i$ and $\text{Var } p_i$ should both be reduced relative to the $\phi_3 = 0$ case. On the other hand, $S$ is positively correlated when $\phi_3 > 0$, tending to increase $VS_i$ and $\text{Var } p_i$. On balance, the first factor of $EV_i$ appears little affected by $\phi_3$ and the second factor appears to exhibit the same qualitative features (e.g., $it \to 1$ as $\pi \to 1$, and $it \to \infty$ as $\pi \to 0$) for any $\phi_3$.

We conclude that given the news process defined above, $EV_i$ is an increasing function of news lumpiness. For $N$ and $\pi t$ large, we have $EV_i \approx k[(1-(1-\pi)^{t+1})/(1-(1-\pi)^{Nt+1})]$, where $k$ depends on $\phi$, and the news distribution $G$. For $\phi_3$ small, $k \approx 1.57$. In particular, $EV_i \to \infty$ as $\pi \to 0$, i.e., as news becomes extremely lumpy.
Lemma 5.1. $E(\Delta f_{t+1} | f_t)$ is a decreasing linear function of $f_t$ whose sign agrees with that of $\bar{X} - f_t$.

Proof: $f_{t+1} = \phi^*(z_{1t}, \ldots, z_{mt}, z_{1t+1}, \ldots, z_{mt+1})$

$$= (1 - \alpha) f_t + \alpha \phi^*(z_{1t+1}, \ldots, z_{mt+1}),$$

since the first equality restates the definition of $f_t$ and the second holds by (B2b) and induction, for some $\alpha \in (0, 1)$. Consequently

$$\Delta f_{t+1} = -\alpha f_t + \alpha \phi^*(z_{1t+1}, \ldots, z_{mt+1}).$$

Now by (B2a), $E(\phi^*(z_{1t+1}, \ldots, z_{mt+1}) | \bar{X}) = \bar{X}$. Since $z_{1t+1}$ is independent of $f_t$, we conclude that $QED$

Lemma 5.2. Assume PPI holds. Then $E(\Delta \nu_{it+1} | \bar{x}_t, p_t)$ is a decreasing function of $\bar{x}_t$ whose sign agrees with that of $\phi^*(z_t) - \bar{x}_t$. Given $\bar{x}_t$, it is independent of $p_t$.

Proof: For fixed $\gamma_{it}$ and $p_t$, we see that $\nu_{it}$ is predetermined and $\nu_{it+1}$ depends only on $z = z_{it}$. More specifically, let $\hat{\gamma}(y) = \psi(\gamma_{it}, 0, y - p_t)$ and define $h(y, z) = \phi(\hat{\gamma}(y), z, 0) - y$. Then $h(\nu_{it}, 0) = 0$ and $h(\nu_{it+1}, z) = 0$ by Proposition 1, PPI and Proposition 2. Applying the implicit function theorem to $h$ and noting assumption (B2b) we see that $\Delta \nu_{it+1} - \nu_{it}$ is a strictly increasing and sign-preserving function of $\phi^*(z_t) - \bar{x}_t$. Since (the response to) $z$ is independent of previous news, we conclude (upon taking expectation over all news paths that yield the given $\gamma_{it}$ and $p_t$) that $E(\Delta \nu_{it+1} | \bar{x}_t, p_t)$ inherits the same properties. The independence from $p_t$ follows from Proposition 2. $QED$

Proof of Proposition 5. $E(\Delta \nu_{it+1} | p_t) = E[E(\Delta \nu_{it+1} | \bar{x}_t, p_t)]$; since $\bar{x}_t$ is monotone increasing in $p_t$, we conclude from the previous lemma that $E(\Delta \nu_{it+1} | p_t)$ is decreasing in $p_t$. Now $\Delta p_{t+1} = \Delta \nu_{it+1}$ if $[i = (M)]$ at $t$ and $t + 1$ ["Event 0"], and is an increasing function of $\Delta \nu_{it+1}$ for $[i = (M \pm k)]$ at $t$ and $i = (M)$ at $t + 1$ ["Event $\pm k"].$ Hence, in any event, $E(\Delta p_{t+1} | p_t)$ is decreasing in $p_t$. Assume for the moment that event 0 occurs, i.e., $\Delta p_{t+1} = \Delta \nu_{(M)1}$. By Proposition 3 we have $\nu_{(M)1} = \bar{x}_{(M)1} = p_t$. Consider the following events:

$$(0) \ ((M)_{i} = (M)_{i+1}), \text{ i.e. the same trader made the } M^{th} \text{ highest bid in period } t+1 \text{ as in period } t;$$

Proof: Suppose to begin with that no trader receives news during period $t$ and the observed price change was $\Delta p_t > 0$. As a consequence of Proposition 2 we then have $\Delta \nu_{it+1} > 0$ for all traders $i$. Consider the following events:
(± k) [(M ± k), = (M),+,1], i.e., the trader who made the (M ± k)th highest bid in period t makes the Mth highest bid in period t + 1, 0 < k < M.

All these events have positive probability since π < 1. Events ± k produce transactions and so involve an announced price change, while with positive probability event 0 does not involve a transaction and hence no announced price change. See Figure 5 for a diagram. Since Δv_t+1 > 0 for all i implies Δp_t+1 ≥ 0 in any event, we see that under present assumptions Δp_t+1 is a non-negative random variable with positive expectation. Moreover, since the magnitude of Δv_t+1 as well as the probabilities of events (± k) are increasing in Δp_t, we conclude that E(Δp_t+1 | Δp_t) is positive and increasing in Δp_t. Likewise, if Δp_t < 0 we find that E(Δp_t+1 | Δp_t) is negative and decreasing in |Δp_t|. Consequently E(Δp_t+1 | Δp_t) is a monotone and sign-preserving function of Δp_t. Since Δf_t+1 = 0 when no news arrives, we conclude that E(Δb_t+1 | Δp_t) has the same property.

Consider now the possibility of private news arrival when f_t = X. It is not hard to see that E(Δb_t+1 | Δp_t) = 0 for i = (M),+,1 in this case, so E(Δp_t+1 | Δp_t) is still an increasing, sign-preserving function of Δp_t, even though Δp_t can differ in sign from Δp_t with positive probability. Again E(Δf_t+1) = 0 — see Lemma 5.1 in the Appendix — so E(Δb_t+1 | Δp_t) remains monotone and sign-preserving in Δp_t. Finally, a continuity argument shows that E(Δb_t+1 | Δp_t) is monotone (though not necessarily sign-preserving) when | f_t - X | is small. QED

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