# Conspicuous Consumption Dynamics *

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**Abstract**

We formalize Veblen's idea of conspicuous consumption as two alternative forms of interdependent preferences, dubbed envy and pride. Agents adjust consumption patterns gradually, in the direction of increasing utility. From an arbitrary initial state, the distribution of consumption among agents with identical preferences converges to a unique equilibrium distribution. When pride is stronger, the equilibrium distribution has a right-skewed density. When envy is stronger, the equilibrium is concentrated at a single point, and the adjustment dynamics involve a shock wave that can be interpreted as a growing, moving, homogeneous "middle class."

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**JEL codes:** C73, D11

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"Since consumption of these more excellent goods is an evidence of wealth, it becomes honorific; and conversely, the failure to consume in due quantity and quality becomes a mark of inferiority and demerit."–Thorstein Veblen (1899, p. 64)

# 1 Introduction

Perhaps taking a cue from John Rea (1834), Veblen popularized the idea of conspicuous consumption: some goods and services, such as expensive interior decorations and lavish parties, are consumed mainly to impress other people. Modern examples include seldom-used second homes, giant motor vehicles and obscure vintages of wine. The consumer's utility arises largely from how these goods rank relative to those consumed by peers: owning a Hummer loses its cachet when everyone else does too.

Conspicuous consumption fascinates many economists and yet remains outside the mainstream because it is not easily captured in standard models of preference, and because it creates a form of interdependence not mediated by the price system. That is, unlike ordinary consumption, conspicuous consumption has a nonpecuniary negative externality: increasing my own rank decreases the ranks of the people I pass (e.g., Akerlof, 1997).

The present paper takes as given that people care to some degree about conspicuous consumption, and examines some of the consequences. It first points to several strands of relevant literature. Then it presents a simple explicit model of preferences for conspicuous and ordinary consumption, and distinguishes two main cases. *Envy* refers to the case where I am more upset by a shortfall of my own conspicuous consumption than I am pleased by an excess in comparison to a peer. *Pride* refers to the opposite case, where my excess matters more.

The next step is characterize equilibrium consumption patterns. In a static game in normal form, a continuum of identical consumers allocates income between ordinary and conspicuous consumption. Proposition 1 shows that in the case of pride, the Nash equibrium choices have a particular dispersed distribution. The distribution is skewed: the density increases at higher levels of conspicuous consumption. Of course, due to the negative externality, the equilibrium distribution typically is Pareto inefficient.

Proposition 2 shows that in the case of envy, there is a continuum of Nash equilibrum distributions, each of which is symmetric. That is, with identical preferences but initial differences, everyone adopts exactly the same consumption pattern, a particular mix of conspicuous and ordinary consumption, but that mix is indeterminate within a range.

We then specify a dynamic adjustment process in which consumers continually adjust consumption in the direction that increases utility, at a rate commensurate with the potential gain. Proposition 3 shows that, in the case of pride, this gradient process converges monotonically to the dispersed Nash equilibrium. The equilibrium distribution emerges asymptotically from any inital distribution, even from one that is completely concentrated. Thus the dynamics can exhibit spontaneous symmetry breaking.

Proposition 4 shows that, in the case of envy, the initial distribution determines the particular Nash equilibrium to which the adjustment process converges. Proposition 5 shows that the adjustment process for envy is non-trivial. From a wide class of smooth initial conditions the dynamics involve compressive shock waves: a moving homogenous clump of consumers (interpreted as a "middle class") emerges in finite time and absorbs an increasing share of the population.

A concluding discussion points out connections with the technical literatures in fluid dynamics and evolutionary game theory. It also mentions possible extensions and other applications of the techniques. Proofs and computational details are collected in an Appendix.

The paper makes two general contributions to existing literature. The first is to demonstrate the surprisingly different consequences of pride and envy. The second is to illustrate a particular sort of adjustment dynamics. We show that Nash equilibria of associated consumer choice (or location) games have non-obvious stability properties, and that transient behavior can have surprising regularities.

# 2 Previous Literatures

Why might people care about how their own consumption patterns rank relative to others'? For suppliers of experience goods such as lawyers and realtors, one possible answer is that conspicuous consumption signals a high income (e.g., Bagwell and Bernheim, 1996), which in turn signals high quality goods. For other people, conspicuous consumption seems to arise from a generalized status seeking motive. One can argue that early humans who preferred to pursue status (at moderate cost) tended to gain relative to those who did not, and their preferences proliferated (e.g., Robson, 1992; Cole, Malaith and Postlewaite, 1992; Samuelson, 2004).

Veblen's ideas have entered the mainstream of sociology. In particular, the theory of relative deprivation (e.g., Davis, 1959; Runciman, 1966) postulates that a person is unhappy when his peers have more of some desireable good. Yitzhaki (1979) connects that theory to the Lorenz curve and the Gini coefficient.

Recently economists have put relative concerns into formal preference models. Fehr and Schmidt (1999) is the most prominent example. It postulates that people dislike income inequality, especially when their own income falls short. Pages 127-128 of Deaton's (2003) survey shows that the Fehr-Schmidt model takes the same form as our equation (2) below when all consumption is conspicuous, after taking expectations across the current consumer population.

Our model also intersects two earlier strands of economics literature.[1] The "keeping up with the Joneses" strand begins with Duesenberry (1949) and includes Pollack (1976), Hirsch (1976), Abel (1990), Cambell and Cochrane (1999), and Ljundqvist and Uhlig (2000), among others. It postulates that people care about the average level of consumption as well as their own personal level. The "rank-dependent utility" strand begins with Frank (1985), and includes Hopkins and Kornienko (2004), and Becker et al. (2005), among others. It postulates that people are concerned with the ordinal rank of their own consumption in the population distribution. Our model is based on cardinal consumption differences between an individual and everyone else in the population. It turns out, however, that the ordinal rank enters key expressions and also that a special case reduces to "keeping up with the Joneses." Numerical examples included in the Appendix show how the models differ.

[1] I am indebted to an anonymous referee for clarifying the connections.

# 3 Preferences for Conspicuous Consumption

We introduce utility functions $\phi$ that capture directly the tradeoff between conspicuous consumption and ordinary consumption. Presumably consumers efficiently allocate expenditures within each of the two categories; our focus is the allocation across categories. Thus let $x$ in [0, 1] denote the share of a consumer's income devoted to ordinary consumption, so $1 - x$ denotes the share devoted to conspicuous consumption.[2]

The consumer receives direct utility $cu(x)$ from ordinary consumption, where the weight $c \geq 0$ is a fixed parameter and $u$ has the classic property of diminishing marginal utility ($u''(x) < 0 < u'(x) \quad \forall x > 0$). Sometimes we also assume that ordinary consumption is a necessity ($u'(x) \to \infty$ as $x \to 0$). A classic example is $u(x) = \ln x$. Sometimes we instead assume that $u \in C^2[0, 1]$, i.e., $u$ is twice continuously differentiable on the entire interval $[0, 1]$.

The consumer also receives utility $r$ from the conspicuous consumption share $(1 - x)$. One simple specification, which we call pure envy, is that the individual suffers to the extent that her own conspicuous consumption falls short of others'. If everyone else chose ordinary consumption share $y$, then the specification would be $r(x, y) = \min\{0, (1 - x) - (1 - y)\} = \min\{0, y - x\}$. If others' choices $y$ have cumulative distribution function $F$, then the payoff is the expected shortfall $r(x, F) = \int_0^x (y - x)dF(y)$. Integrating by parts, we can write this as $r(x, F) = -\int_0^x F(y)dy$.[3] Putting together the two components $r$ and $cu$, we obtain the pure envy utility function

$$\phi^E(x, F) = cu(x) - \int_0^x F(y)dy. \qquad (1)$$

At the other extreme, relative concerns may take the form of pure pride: the consumer is pleased to the extent that her own conspicuous consumption exceeds that of others. Now $r(x, F) = \int_x^1 (y - x)dF(y)$. More generally, with non-negative weights $a$ and $b$ respectively on envy and pride, we have $r(x, F) = a \int_0^x (y - x)dF(y) + b \int_x^1 (y - x)dF(y)$, or, integrating

[2] Apologies to readers who would prefer that $x$ denote the conspicuous consumption share. Unfortunately that convention would clash with the standard convention that cumulative distribution functions are right-continuous and denote the fraction of choices less than (not greater than) $x$.

[3] This expression differs from Yitzhaki's (1979) "relative satisfaction" by only an additive term $x$, and likewise the expression below for pure pride is essentially the same as Yitzhaki's "relative deprivation."

by parts, $r(x, F) = b(1 - x - \int_0^1 F(y)dy) + (b - a)\int_0^x F(y)dy$. The overall utility function then is

$$\phi(x, F) = cu(x) + b(1 - x - \alpha_F) + (b - a)\int_0^x F(y)dy, \tag{2}$$

where $\alpha_F = \int_0^1 F(y)dy \in [0, 1]$.

The parameter $c$ represents importance of ordinary consumption relative to conspicuous consumption. Note that, as long as $c > 0$, the consumer will choose $x > 0$, i.e., at least some ordinary consumption, because it is a necessity (i.e., $u'$ is very large when $x$ is very small). We will soon see the qualitative behavior of the model hinges on the sign of $b - a$, i.e., on whether envy or pride is the stronger motive. We will cover both cases, but the literature on loss aversion suggests to us that envy is generally stronger.[4]

Equations (1) and (2) indicate that a consumer's current utility depends on the current distribution $F$ of choices across the population of consumers as well as the consumer's current choice $x$. For simplicity, assume that everyone has the same preferences and the same income. Assume also that the population is large, so the consumer's own choice has a negligible impact on the cumulative distribution $F$. Holding constant others' choices $F$, then, each consumer faces the same *utility landscape* described by the graph of $\phi(\cdot, F)$, i.e., the graph of the function $x \mapsto \phi(x, F)$ giving her payoff as a function of her own choice. Figure 1 shows landscapes for $c = 0.1$ and $u(x) = \ln x$ given two different distributions $F$.

Figure 1: Two landscapes on the unit interval. The utility function is (2) with $u = \ln$, $a = 1, b = 0$, and $c = 0.1$. The distribution $F$ is uniform on [0,1] in Panel A and is uniform on [.9, 1] in Panel B.

[4]Loss aversion is the first general psychological tendency mentioned in the classic survey of Rabin (1998). The idea — that relative to some "reference level" a loss or shortfall hurts more than the same size excess helps — is more often applied to comparisons across time for a single consumer, but Rabin notes implicitly (e.g., in his intial cite of Duesenberry) that it also applies to interpersonal comparisons at a single point of time.

Figure 2: The equilibrium cumulative distribution function for (3) with c=0.2.

# 4 Equilibrium

Intuitively, a landscape is in equilibrium if the entire population lives only on its high points (or plateaus). In the next section we will formalize the idea dynamically: people adjust their consumption patterns in order to increase utility, so the landscape and the population distribution coevolve until, perhaps, they reach a steady state equilibrium. In this section, we formalize the intuition using the static notion of Nash equilibrium.

## 4.1 The location game.

Consider a continuum of players indexed $i \in I = [0, 1]$, each of whom simultaneously chooses a point $X(i)$ in the strategy set $S_i = A = [0, 1]$. Thus a *strategy profile* is a Lebesgue measurable map $X : I \to A$. The payoff function is $\phi(X(i), F)$, where $\phi$ is defined in equation (2) and $F$ is the cumulative distribution function for the strategy profile $X$, i.e., $F(x)$ is the Lebesgue measure of $\{i \in I : X(i) \leq x\}$.

The strategy profile $X$ is a *Nash equilibrium* if there is no player $i \in I$ and strategy $y \in A$ such that $\phi(X(i), F) < \phi(y, F)$. The profile $X$ is *efficient* if there is no alternative profile $Y$ that provides at least as high a payoff for every player, and a strictly higher payoff for a subset of players of positive measure.

By definition, at Nash equilibrium any consumption pattern $x$ chosen by any player maximizes utility. The associated first order condition is $\phi_x(x, F) = 0$ for $x \in (0, 1)$, with the usual Kuhn-Tucker inequalities at the boundary points $\{0, 1\}$. Thus the equilibrium analysis (and, it turns out, also the dynamic analysis) depends on the utility function only via its gradient $\phi_x(x, F)$.

## 4.2 Pride Equilibrium

To understand the consequences, consider first the case of pure pride and ordinary log utility, $\phi(x, F) = c \ln x + \int_x^1 (y - x) dF(y)$ with gradient

$$\phi_x^P = c/x - 1 + F(x). \tag{3}$$

This gradient is positive for $x < c$, so consumers will want to increase consumption of ordinary goods until $x \geq c$ or until they hit the upper boundary $x = 1$. Thus in a distribution $F^*$ corresponding to a Nash equilibrium profile $X^*$ we must have $F^*(x) = 0$ for $x < \min\{c, 1\}$.

Since short-selling is not allowed for either ordinary or conspicuous consumption, we must have $F^*(0) = 0$ and $F^*(1) = 1$. There can be a mass point at $x = 1$, i.e., in Nash equilibrium, a positive fraction of the population might refrain altogether from conspicuous consumption. Indeed, if $c \geq 1$, then nobody will indulge in conspicuous consumption, so the rest of this discussion assumes $c < 1$.

The other consumers choose equilibrium allocations $x \in [c, 1)$ such that the gradient vanishes, i.e., such that $c/x - 1 + F^*(x) = 0$. Thus for $0 < c < 1$ the equilibrium cumulative distribution function $F^*(x)$ follows the hyperbolic arc $1 - c/x$ inside the unit square $(x, F^*) \in [0, 1] \times [0, 1]$. The equilibrium distribution follows the lower edge $F^*(x) = 0$ when the arc is below the square, and jumps to the upper right corner when the arc intersects the right edge, as in Figure 2.

In Nash equilibrium all consumers attain the same utility, which direct calculation (see Appendix) shows to be 0. Within the support $[c, 1]$ of the equilibrium distribution $F^*$, the marginal gain $1 - F^*(x)$ from increasing conspicuous consumption is exactly offset by the corresponding marginal cost $-c/x$ of foregone ordinary consumption. Utility $c \ln(x/c) - c + x$ is lower in the unpopulated zone $x \in [0, c)$. Perhaps surprisingly, the equilibrium is efficient: the highest possible utility occurs when everyone refrains from conspicuous consumption, but that utility is still zero.

For the more general case (2), the gradient is

$$\phi_x = cu'(x) - b + (b - a)F(x). \tag{4}$$

As long as pride is stronger than envy, Nash equilibrium resembles that of the example except that it is generally inefficient. The formal result is:

**Proposition 1.** Suppose $b > a \geq 0$ and $c \geq 0$ in the utility function (2). Then there is a unique cumulative distribution function $F^*(x)$ such that

1. every Nash equilibrium of the location game has cumulative distribution function $F^*$;

2. the density $f^* = F^*_x$ exists and is a continuous, strictly decreasing function on the interior of the support $[x_o, x_1]$ of $F^*$, where $0 \leq x_o \leq x_1 \leq 1$;

3. when $x_1 = 1$, the equilibrium distribution can have a jump discontinuity at that point; and

4. every Nash equilibrium with some conspicuous consumption is inefficient iff $a > 0$.

Proofs of all Propositions appear in the Appendix. The logic behind the proposition is essentially the same as in the example: the gradient (4) is zero along a curve $\hat{F}(x) = \frac{b}{b-a} - \frac{c}{b-a}u'(x)$ and $F^*$ is $\hat{F}$ truncated by the unit square $0 \leq x, F^* \leq 1$. The density is decreasing because ordinary consumption provides decreasing marginal utility. The proof also shows that inefficiency in equilibrium arises entirely from envy, while the effects of pride exactly offset.

### 4.3 Envy Equilibrium

The preceding analysis breaks down in the arguably more important case $a > b \geq 0$, where envy is stronger than pride. To see how, consider the pure envy case (1) with log ordinary utility and weight parameter $c$ strictly between 0 and 1. At an arbitrary continuous distribution $F$, the gradient is $\phi_x = c/x - F(x)$, which is strictly decreasing in $x$, positive near $x = 0$ and negative at $x = 1$. Hence the gradient is zero at a single point $\widetilde{x} \in (0, 1)$, positive for $x \in (0, \widetilde{x})$ and negative for $x \in (\widetilde{x}, 1]$. Clearly the underlying strategy profile is not Nash; only the player choosing $\widetilde{x}$ is maximizing. In this simple example, then, there can be no Nash equilibrium distribution with a proper density function. It turns out that any Nash equilibrium must be symmetric: everyone must choose the same consumption pattern $\widetilde{x}$.

The formal result is written using the Heaviside notation $H(z) = 1, z \geq 0;\ H(z) = 0, z < 0$. Thus $H(x - x^*)$ has a maximal jump discontinuity at $x = x^*$ and so represents a symmetric strategy profile in which almost everyone chooses the same allocation $x^*$. The result also refers to points $0 \leq x_o \leq x_1 \leq 1$; as in the previous Proposition, the points satisfy $u'(x_o) = b/c$ and $u'(x_1) = a/c$ where these expressions are well defined.[5]

**Proposition 2.** Suppose $a > b \geq 0$ and $c \geq 0$ in the utility function (2). Then

1. every Nash equilibrium of the location game has a cumulative distribution function of the form $H(x - x^*)$, for some $x^* \in [x_o, x_1]$;

2. conversely, if $x^* \in [x_o, x_1]$ then $H(x - x^*)$ is the cumulative distribution of a Nash equilibrium; and

3. every Nash equilibrium with some conspicuous consumption is inefficient.

What does Nash equilibrium look like in the special case $a = b > 0$? After reading the proofs of Propositions 1 and 2, the reader can easily verify that then it is symmetric ($F^*(x) = H(x-x^*)$ as in Proposition 2 but at the same time it is a degenerate version of the equilibrium in Proposition 1, with $x^* = x_o = x_1$ and this value of $x$ satisfies $cu'(x) = a = b$. Consequently at this equilibrium, as in "keeping up with the Joneses," consumers gain higher utility when they make choices closer to the average choice $x^*$.

# 5 Adjustment Dynamics

The static analysis leaves open some important questions. In the general envy case, there is a continuum of Nash equilibrium distributions. Which (if any) is most likely to be observed? In the general pride case, there is a unique Nash equilibrium distribution, but can it predict the behavior of a large group of similar consumers? Occasionally the available consumer goods will change, and the preference parameters $a, b, c$ might shift. Can we predict the impact? Static equilibrium theory provides no natural way to answer such questions.

Dynamics enrich the model and offer predictions of transient as well as equilibrium (steady state) behavior. Steady states that are locally (or, better, globally) asymptotically stable are much more plausible predictions of long run behavior than are other equilibria. When initial conditions affect asymptotic behavior we have a way to select among multiple equilibria,

[5]Otherwise, as detailed in the first paragraph of the proof of Proposition 1, the points are 0 or 1.

and, given reasonable adjustment rates, these dynamic considerations provide a basis for comparative statics predictions.

Our dynamics arise from the following behavioral assumption: each consumer responds to the current utility landscape by adjusting $x$ continuously so as to increase utility. There are at least three reasons to assume that consumption adjusts continuously rather than instantaneously. First, faster adjustment is generally more expensive. We don't explicitly model adjustment cost here, but note that if cost is quadratic in the speed of adjustment, then gradient dynamics, defined below, arise as the efficient tradeoff between utility and adjustment cost (Friedman and Yellin, 1997, Proposition 1). Second, even if a consumer's conspicuous consumption takes a large jump, there may be a perceptual lag for other consumers, so that pride and envy adjust more gradually. Third, not all consumers at the same allocation $x$ will adjust at exactly the same time. Their average adjustment will be more gradual and systematic, and that is what the model actually analyzes.

Continuous adjustment in the direction of increasing utility is captured in gradient dynamics: the rate at which a consumer adjusts the allocation $x$ is proportional to the slope of the landscape. That is, scaling the proportionality constant to 1.0, a consumer adjusts his choice $x$ over time at rate $dx/dt = \phi_x(x, F) \equiv \partial\phi(x, F)/\partial x$.[6]

Of course, as all individual consumers adjust, the distribution of their choices also changes. Let $F(x, t)$ denote the cumulative distribution function of consumer choices at time $t \geq 0$, i.e., the current fraction of the population devoting no more than $x$ to ordinary consumption. That population fraction increases over a short period of time to the extent that (a) local consumers are numerous and (b) they rapidly decrease ordinary consumption. More precisely, $F_t(x, t) \equiv \partial F(x, t)/\partial t$ is positive to the extent that (a) the population density $F_x(x, t) = f(x, t)$ is large and (b) the adjustment velocity $dx/dt = \phi_x(x, F)$ is negative and has large absolute value. Thus (at all points $x \in (0, 1)$ of absolute continuity), gradient dynamics are described by the master equation

$$F_t(x, t) = -F_x(x, t)\phi_x(x, F). \tag{5}$$

[6]Gradient dynamics are standard in biology (e.g., Lande, 1982) and in machine learning (e.g., Bishop, 1995, sec. 7.5), as well as in fluid dynamics and other physical systems (e.g., Whitham, 1974). Economists also occasionally turn to gradient dynamics when considering gradual adjustment (e.g., Sonnenschein, 1982).

Since short-selling is not allowed for either ordinary or conspicuous consumption, there are boundary conditions at $x = 1$ and $x = 0$. As shown in the Appendix, these conditions allow population mass to accumulate at either endpoint.

### 5.1 Pride Steady State

In steady state equilibrium, the distribution $F(x,t)$ is constant over time and so its time derivative $F_t(x,t)$ is 0. A glance at the master equation (5) indicates that the key condition is that the gradient $\phi_x(x,F) = 0$ at all choices $x$ currently used, as in the static Nash equilibrium analysis.

In the case of pride, it turns out that the master equation equation has a unique steady state, and that it is precisely the Nash equilibrium distribution. Equally important, the gradient dynamics are well defined from any initial distribution function, and they converge monotonically to that equilibrium distribution. Thus the dynamic theory justifies comparative statics predictions on the basis of a global stability argument, and rules out any complex transient behavior. The formal result is:

**Proposition 3.** Suppose $u \in C^2[0,1]$, $b > a \geq 0$ and $c \geq 0$ in the utility function (2), and let $F_o(x) = F(x,0)$ be an arbitrary initial cumulative distribution function. Then,

1. there is a unique solution $F(x,t)$ to the general master equation for all $t > 0$;
2. $F(x,t)$ converges as $t \to \infty$ to the distribution function $F^*(x)$ given in Proposition 1;
3. The $L^2$-distance between $F(x,t)$ and $F^*(x)$ decreases in $t$ to zero.

The asymptotic distribution $F^*(x)$ is dispersed, except perhaps at $x = 1$. The reader might wonder what happens if some or all of the population initially is clumped at some consumption pattern $x < 1$. As explained in the Appendix, it turns out that this initial clump is immediately dispersed in the form of a rarefaction fan (sometimes called a rarefaction wave).

## 5.2 Envy Steady State

A dynamic analysis shows why the pride equilibrium breaks down when envy is the stronger motive. The master equation (5) allows only three possible sorts of choices $x$ in steady state $F^*$. Since we must have $F_t^*(x) = 0$ everywhere the master equation is well defined, either

1. $F_x^*(x)$ is undefined because $x$ is a jump discontinuity, i.e., a mass point; or
2. $F_x^*(x) = 0$, i.e., the density is zero at $x$; or
3. $\phi_x = 0$, i.e., the gradient is zero at $x$.

The dispersed equilibrium distribution for pride arose from the last possibility: the gradient was zero over a non-trivial interval. But we saw in the previous section that this can't happen when $a > b \geq 0$. Then there is some $\widetilde{x} \in [0, 1]$ such that the gradient is positive for $x \in [0, \widetilde{x})$ and is negative for $x \in (\widetilde{x}, 1]$. The master equation now tells us that consumers move towards $\widetilde{x}$ from both directions until the entire population has clumped together at $\widetilde{x}$.

The next proposition states this formally and notes that it is the initial distribution that determines the symmetric equilibrium choice $x^* = \widetilde{x}$. It requires a bit of special notation to characterize that choice when the initial distribution has jump discontinuities. For an arbitrary distribution $F$, let $S_F = \{x \in [0, 1] : cu'(x) - b > (a - b)F(x)\}$ and let $y_F = \sup S_F$ if $S_F$ is nonempty, and $y_F = 0$ otherwise.

**Proposition 4.** Suppose $u \in C^2[0, 1]$, $a > b \geq 0$ and $c \geq 0$ in the utility function (2), and let $F_o(x) = F(x, 0)$ be an arbitrary initial cumulative distribution function. Then

1. there is a unique solution $F(x, t)$ to the master equation (5) for all $t > 0$;
2. $F(x, t)$ converges as $t \to \infty$ to an equilibrium cumulative distribution function $F^*(x)$; and
3. $F^*(x) = H(x - \widetilde{x})$, where $\widetilde{x} = y_{F_o} \in [x_o, x_1]$.

### 5.3 Envy Dynamics and a Homogeneous "Middle Class"

The transient dynamics for envy are especially interesting. To illustrate, suppose $c = 0$ in the pure envy case, and consider the unimodal initial density $f(x, 0) = 6x(1 - x)$, with corresponding cumulative distribution function $F(x, 0) = 3x^2 - 2x^3$. The Appendix shows that the master equation has an analytic solution $F(x, t)$ with a continuous unimodal density $f(x, t)$ for $t \in [0, 2/3)$. The mode (or peak) $x^*(t)$ decreases steadily from 1/2 at $t = 0$ to 1/6 at $t = 2/3$, while the height of the mode becomes unbounded as $t \to 2/3$. The intuition is that all consumers decrease ordinary consumption (recall that for $c = 0$ they care only about conspicuous consumption) but gradient dynamics dictate that the modal consumer adjusts more rapidly than consumers with initially lower $x$ and he begins to overtake them at time $t^* = 2/3$ and consumption level $x = 1/6$. Given our assumption of identical underlying preferences and income, this consumer can't actually pass his rivals because his behavior is identical to theirs once he attains the same consumption level. Instead, he clumps together with them, and the clump grows as it overtakes consumers with $x$ just below the mode and is overtaken by consumers with $x$ just above the mode.

Thus, beginning at $t = 2/3$ we get a growing, moving mass of consumers with identical consumption patterns, a homogeneous "middle class."[7] See Figure 3. The Appendix shows how to compute its position $x^*(t)$ and mass $M(t)$ for $t > 2/3$. These are incorporated in a distribution function $F(x, t)$ with a jump discontinuity. It turns out that for $t \in (2/3, 1]$ the position is $x^*(t) = (1 - t)/2$ and the jump size is $M(t) = \sqrt{3t^{-2} - 2t^{-3}}$. Thus the middle class absorbs the entire population by the time it hits the boundary $x = 0$ at time $t = 1$. For $t > 1$, of course, everyone continues to neglect ordinary consumption.

So much for the numerical example. What can one say for more general initial distributions and positive weight $c$ on ordinary consumption? The next result guarantees very similar behavior when $c$ is small.

**Proposition 5.** Suppose $c \geq 0$ in the utility function (1), and let the initial distribution $F_o(x)$ be thrice continuously differentiable, with density $f$ that attains a regular maximum at $x = q \in (0, 1)$. Then for all sufficiently small $c$, the solution of (5) has an interior compressive

[7]This label is loose, intended only to connote a subpopulation moving together in the interior of the population distribution. It does not correspond precisely to any sociological definition of the middle class.

Figure 3: Emergence of a "middle class." The cumulative distribution function $F$ is shown for time $t = 0$ and subsequent times. At $t^* = 2/3$ the distribution has a vertical tangent at $x^* = 1/6$. At later times the distribution has a jump discontinuity. The dotted red line at $t = 4/5$ extrapolates the simple analytic solution (19) obtained in the Appendix. Since it is multiple valued, it must be trucated using the equal area condition (26) to obtain the true (discontinuous) distribution function.

shock wave. Up to first order in $c$, the shock wave emerges at time

$$t^*(c) = 1/f(q) + cT(q) + O(c^2)$$

and location

$$x^*(c) = q - F(q)/f(q) + c\hat{X}(q) + O(c^2) \in (0, 1).$$

The proof in the Appendix includes formulas for the functions $T(q)$ and $\hat{X}(q)$. Numerical explorations suggest that shock waves will arise from general utility functions with $a > b \geq 0$ and weights $c$ not very close to zero, as long as there are local maxima of the initial density far enough above the point $\widetilde{x}$ where the gradient is 0. However, it seems difficult to derive analytic expressions in the more general cases.

# 6 Discussion

The main conclusions can be summarized as follows. When pride (enjoyment of excess conspicuous consumption) is a stronger motive than envy (chagrin at shortfalls), other things equal, consumers will tend to disperse into a range of consumption patterns following a particular skewed distribution. At the margin, each consumer's benefit from moving up the ranks of conspicuous consumers is exactly offset by his cost in foregone ordinary consumption. The equilibrium is unique and globally stable: under gradient adjustment dynamics, the population distribution relaxes towards this particular distribution from any initial consumption pattern.

In the arguably more important case when envy is stronger than pride, all consumers with the same income and preferences will tend to all consume conspicuously in the same proportion. If there are initial differences, they are absorbed in a moving, growing "middle class," represented mathematically as a moving point mass. The game has multiple equilibria, and gradient dynamics select a particular equilibrium distribution defined by the initial consumption pattern.

In both cases, as long as envy is present to some degree, the final consumption patterns are inefficient. All consumers would achieve higher utility if everyone proportionately reduced conspicuous consumption.

The analysis relies on mathematical techniques that come mainly from evolutionary game theory and from fluid dynamics. In the language of evolutionary games (e.g., Weibull, 1995; Sandholm, 2005), the utility function $\phi$ is a payoff or fitness function that depends on the player's own strategy $x$ and the current state $s$. In a typical evolutionary game, the set $A$ of admissible strategies is finite and unordered, e.g., $A = \{a_1, ..., a_K\}$, and the state $s$ is the current distribution of strategies in the population, e.g., a point in the $K$- simplex. Our analysis of conspicuous consumption instead takes strategies $x$ in the ordered interval $A = [0, 1]$, so the state $s = F$ is a point in $\mathcal{F}$, the infinite dimensional simplex of distribution functions (or probability measures).

Typical evolutionary game dynamics are assumed to be monotone in payoffs, e.g., the growth rates $\dot{s}_k/s_k$ of population shares $k = 1, ..., K$ have the same ordering as the corresponding fitnesses $\phi(a_k, s)$. The leading example is replicator dynamics, where $\dot{s}_k = s_k(\phi(a_k, s) - \bar{\phi}(s))$, i.e., each growth rate is equal to its fitness relative to average fitness $\bar{\phi}(s) = \sum_k s_k \phi(a_k, s)$. Our conspicuous consumption dynamics are monotone in a different sense: each individual adjusts locally by moving up the fitness (i.e., utility) gradient. This feature allows us to revive and extend the landscape metaphor of Sewall Wright (1949).[8] Wright described static fitness landscapes for continuous traits. Our utility landscape is dynamic: it morphs as the distribution $F$ changes over time. Friedman and Yellin (1997) show that, consistent with

[8]Several authors including Bomze (1991), Oechssler and Riedel (2001, 2002), and Cressman and Hofbauer (2005), have generalized replicator dynamics to continuous strategy sets $A$. However, these generalizations do not use the intrinsic ordering of $A$ (the labelling is as arbitrary as $k$ is with finite choice sets) and so the dynamics do not lend themselves to a landscape interpretation. The intuition is that in replicator dynamics individuals never adjust. The dynamics describe birth and death rates at different strategies, and new births don't generally occur in the neighborhood of recent deaths.

Propositions 3 and 4 above, equilibrium landscapes have peaks or plateaus at the support of the equilibrium distribution $F^*$.

In general, the dependence of the fitness function $\phi$ on the current state $F$ can take many forms. Perhaps the most obvious form begins with a payoff function $g(x, y)$ from some symmetric two-player game, and takes its expectation $\phi(x, F) = \int g(x, y)dF(y)$. A second form, often used in the biology literature, is to assume that only the mean action $\mu_F = \int ydF(y)$ enters the payoff function. A third simple form, ubiquitious in fluid dynamics, is that $\phi$, or its gradient $\phi_x$, depends on $F$ only via the local value $F(x)$.

This last form is what turns out to be relevant for the conspicuous consumption model, as can be seen from equation (4). Consequently the master equation (5) takes the same form as in fluid dynamics. The consumer population mass turns out to be analogous to a compressible inviscid fluid which can support compressive shocks as in Proposition 5 and rarefaction waves (or symmetry breaking) as mentioned after Proposition 3. The Appendix notes specific connections to the fluid dynamics literature.

The consumption model can be generalized in many respects. First, as suggested at the beginning of Section 3, we could include a little behavioral or perceptual noise. This takes the form of a diffusion term in (5), as is assumed in quantal response equilibrium models; see Anderson, Goeree and Holt (2004) for example. Low amplitude noise would alter the Propositions slightly. It would round off the kinks (where $F^*$ intersects the boundary of the square) in the long run equilibrium described in Proposition 1. The clump in the long run equilibrium in Proposition 2 would be slightly smeared, approximately Gaussian with small variance, and its center $\widetilde{x}$ would lose its dependence on the initial distribution. The shock waves described in Proposition 3 would also be smeared: instead of travelling jump discontinuities there would be travelling steep segments of a continuous distribution, locally approximately Gaussian.

The current version of the model suggests that conspicuous consumption is ultimately fruitless: envy is frustrated in the long run because everyone ends up with the same consumption pattern $\widetilde{x}$, while pride leads to a range over which consumers are indifferent. A more realistic model would include individual differences in the personal tradeoff parameter $c$ for conspicuous and ordinary consumption, as well as income differences. The state then must include the income distribution as well as the choice distribution, and one would have to keep track of several player populations, one for each value of $c$. Such complications seem analytically challenging, but are amenable to simulation methods.

Such an extended model might provide useful policy insights. For example, Pigouvian taxes could be imposed to reduce the external costs of conspicuous consumption, but the equilibrium impact (and transient dynamics) conceivably might suggest unintended consequences. Pride and envy no doubt affect public acceptance of congestion pricing (e.g., for electric power during heat waves, or for highway toll lanes), and an extended model might point to unexpected short run and long run impacts.

Finally, one can also imagine using the methods introduced here in a variety of different social science and biology applications in which adjustment takes place at the individual level and is continuous. Biological applications (apparently not yet fully exploited) might include the evolution of continuous traits such as beak size of finches, or preferred altitude zone. A promising economic application is to financial markets dominated by portfolio managers whose compensation depends on relative performance. Each manager continously adjusts $x$, the portfolio risk or leverage. One can imagine dynamics that sometimes lead to bubbles and crashes. Other natural economic applications include Hotelling-style models of spatial competition in industry and political models in the tradition of Anthony Downs (1957).

# 7 Appendix A: Mathematical Details

Here we assemble the tools needed for our proofs and computations. Evans (1998), Smoller (1994) and Whitham (1974) are excellent background sources.

*The conservation law and its solution.* We first provide more detail on the interpretation of (5), the master equation $F_t(x,t) = -\phi_x(x,F)F_x(x,t)$, given the general form (4) of the gradient $\phi_x(x,F) = cu'(x) - b + (b-a)F(x)$. As in Propositions 3 and 4, we assume here that $u \in C^2[0,1]$. Although technically this rules out classic examples such as $u(x) = \ln(x)$ where $u'(0)$ is undefined or unbounded, it does allow for arbitrarily close approximations to these examples, such as $u(x) = \ln(x+\epsilon)$ where $\epsilon$ is an arbitrarily small positive number.[9]

We shall obtain solutions using the method of vanishing viscosity. The idea is to add a little behavioral or perceptual noise and let its amplitude go to zero. As noted in the text, noise is modeled by adding a diffusion term to the master equation. Specifically, we have that $F^\varepsilon$, the solution to the equation with noise of amplitude $\varepsilon$, satisfies the second order partial differential equation

$$F_t^\varepsilon(x,t) = -\phi_x(x,F^\varepsilon)F_x^\varepsilon(x,t) + \varepsilon F_{xx}^\varepsilon(x,t).$$

Following the fluid dynamics literature, rewrite the equation in the standard form:

$$F_t^\varepsilon(x,t) + \frac{d}{dx}\left[h\left(x,F^\varepsilon(x,t)\right)\right] = g\left(x,F^\varepsilon(x,t)\right) + \varepsilon F_{xx}^\varepsilon(x,t) \tag{6}$$

where the *flux* function, $h$, is

$$h\left(x,F(x,t)\right) = \left(cu'(x) - b\right)F(x,t) + \frac{1}{2}(b-a)F^2(x,t) \tag{7}$$

and the *source* term, $g$, is

$$g\left(x,F(x,t)\right) = cu''(x)F(x,t). \tag{8}$$

It is well known (e.g., Ladyzenskaja and Ural'Ceva, 1965; Friedman, 1969) that, for each $\varepsilon$, there is a unique *classical* solution $F^\varepsilon$ to (6) subject to the initial condition $F^\varepsilon(x,0) = F_0(x)$ and the boundary conditions $F^\varepsilon(0,t) = 0$ and $F^\varepsilon(1,t) = 1$. By "classical solution", we mean that $F^\varepsilon$ is continuous at the initial condition $t = 0$ and $C^2$ smooth for $t > 0$. Note

[9] We conjecture that these approximations could be used to extend the results cited in this section to show that Propositions 3 and 4 hold even when $u'(x) \to \infty$ as $x \to 0$, but we do not pursue the issue further here.

that the boundary conditions are necessary conditions for $F^\varepsilon(\cdot,t)$ to be a smooth cumulative distribution function on $x \in [0,1]$ when $t > 0$.

Bardos et al (1979) shows that $F$, the solution to the master equation without noise, is properly defined as the limit of $F^\varepsilon$ as $\varepsilon$ goes to zero. Specifically, $F^\varepsilon$ converges (in the $L^1$ sense) to a unique function $F$ of bounded variation, which is the unique solution to the Cauchy problem comprised of the master equation, (5), which can be written in the fluid dynamics form

$$F_t(x,t) + \frac{d}{dx}\left[h\left(x, F(x,t)\right)\right] = g\left(x, F(x,t)\right), \tag{9}$$

subject to the initial condition

$$F(x,0) = F_0(x)$$

and the boundary conditions

$$F(0,t) = 0 \text{ and } F(1,t) = 1.$$

In the fluid dynamics literature $F$ is called the "vanishing viscosity" solution, or, equivalently, in this context, the "entropy" solution.

It bears emphasizing that $F$ is a weak solution (see Bardos et al (1979) and the background sources), not necessarily classical. In particular, $F$ can be discontinuous even when the initial condition, $F_0$, is smooth, so the master equation (9) requires reinterpretion. One proceeds as follows. First define the domain $\Omega = \{(x,t) : x \in (0,1),\ t > 0\}$ and then define a test function $\rho(x,t)$ as any $C^1$ smooth function with compact support (that is, the domain where $\rho \neq 0$ is contained within a closed, bounded subset of $\Omega$). If we multiply (9) by any test function $\rho(x,t)$, then integrate over $\Omega$, and finally apply integration by parts to both the $x$ and $t$ derivatives, we obtain

$$-\iint_\Omega F(x,t)\rho_t(x,t) + h\left(x, F(x,t)\right)\rho_x(x,t)dxdt = \iint_\Omega g\left(x, F(x,t)\right)\rho(x,t)dxdt. \tag{10}$$

Note from the compact support of $\rho$, there are no boundary integrals in (10), and note that since $\rho$ is smooth, the derivatives in (10) — unlike the derivatives in (9) — are always defined. A weak solution to the PDE (9) is then defined as a function $F(x,t)$ that satisfies (10) for any test function $\rho$.

From this formulation, it is straightforward (e.g., using the methods of Evans, 1998, p.138-140), to show that if $F$ is discontinuous on a smooth curve $x = s(t)$ within $\Omega$, then the following *Rankine-Hugoniot* jump condition must be satisfied:

$$\frac{ds}{dt} = \frac{h(s(t), F_R) - h(s(t), F_L)}{F_R - F_L}, \tag{11}$$

where $F_L = \lim_{x \nearrow s(t)} F(x,t)$ is the limit as we approach the curve from the left and $F_R = \lim_{x \searrow s(t)} F(x,t)$ is the limit as we approach the curve from the right. The curve, $s(t)$, where $F$ is discontinuous is called a *shock*.

On a shock, we can define $F$ to be either its left or right limit. (This has no effect on (10) since it is known that the set of all shocks forms a set of measure zero in the domain $\Omega$.) Since cumulative distribution functions are conventionally defined to be right continuous, we will define $F$, at any point of discontinuity in $\Omega$, by its right limit. Note, however, that the boundaries, $x = 0$ and $x = 1$, are not in $\Omega$. Therefore, $F$ may not be right continuous at $x = 0$. This is desirable since we need $F$ to be able to indicate a concentration of probability mass if it occurs at $x = 0$. Likewise, we normalize the initial condition, $F_0(x)$, to be right continuous except possibly at $x = 0$.

There is also a continuity question at the boundaries. The approximations $F^\varepsilon$ are continuous (in fact, smooth) at the boundaries, but when is $F$ continuous at its boundaries? To answer this, and later to prove that the solution remains a cumulative distribution function, we will employ a standard tool: *characteristics*. Roughly speaking for the current context, a characteristic curve $\chi(t)$ traces the consumption path followed by a particular individual over time, and one can think of the solution $F(x,t)$ as a labelling of the collection of such curves. A more precise description appears below.

The characteristics for the master equation (9) have particularly nice properties due to the fact that $h_{FF} = b - a$, a constant, so the flux function $h$ is either strictly convex if $b - a > 0$, or strictly concave if $b - a < 0$, or linear if $b - a = 0$. Dafermos (1977) covers the first two cases; the third, linear, case is simpler. In all three cases, there are classical (smooth) characteristic curves associated to the solution that are known to have the following five properties.

1) From each point $(x_o, t_o)$ in $\Omega$, there is a $C^1$ smooth classical characteristic curve, $x = \chi(t)$, that emanates backwards in time from $(x_o, t_o)$ through $\Omega$ until it terminates on either the initial condition $t = 0$ or on one of the boundaries $x = 0$ or $x = 1$.

2) Within $\Omega$, the classical characteristic curve, $\chi(t)$, and the value of the solution on the

classical characteristic, $F(\chi(t),t)$, are continuously differentiable solutions to the ordinary differential equations $\frac{d\chi(t)}{dt} = h_F(\chi(t), F(\chi(t),t))$ and $\frac{dF(\chi(t),t)}{dt} = -h_x(\chi(t), F(\chi(t),t)) + g(\chi(t), F(\chi(t),t))$. Specifically, using the flux function (7) and the source function (8), we have

$$\frac{d\chi(t)}{dt} = \phi_x(\chi, F) = (cu'(\chi(t)) - b) + (b-a)F(\chi(t),t) \tag{12}$$
$$\frac{dF(\chi(t),t)}{dt} = 0, \tag{13}$$

so $F$ remains constant on a classical characteristic, except possibly at its termination point.
3) $F$ also remains constant at the termination point if the termination point is a point of continuity for the initial condition function $F_0(x)$ or, for $t > 0$, if the termination point is on the boundaries $x = 0$ or $x = 1$.
4) The possibilities at a termination point, $x_D$, where the initial condition is discontinuous depend upon the curvature of the flux term. In the linear case, $b-a = 0$, $F$ remains constant at the termination point (i.e., $F = F_0(x_D)$). In the concave case, $b - a < 0$, no classical characteristic can terminate at $x_D$. In fact, since the left hand side of (12) is decreasing in $\chi$, the jump discontinuity can only grow over time., i.e., a shock intersects the initial condition line $t = 0$ at $x = x_D$. In the convex case, $b - a > 0$, the constant value $F$ on the classical characteristic when $t > 0$ can take any value from the set $[F_L, F_R]$ where, if $x_D \in (0,1]$, we let $F_L = \lim_{x\to x_D^-} F_0(x)$ and $F_R = F_0(x_D) = \lim_{x\to x_D^+} F_0(x)$ and if $x_D = 0$, we let $F_L = 0$ and $F_R = \lim_{x\to x_D^+} F_0(x)$. In this case we have a *rarefaction fan* emanating in forward time from the point $(x_D, 0)$; that is, a collection of classical characteristics (one for each $F \in [F_L, F_R]$) that fan out from the point of discontinuity, $(x_D, 0)$, with speeds given by (12).
5) The classical characteristic curves corresponding to two distinct points in $\Omega$ with the same time, namely, $(x_0, t_0)$ and $(x_1, t_0)$ where $x_0 < x_1$, cannot intersect unless the curves belong to the same rarefaction fan, in which case the curves only intersect at their common termination point, $(x_D, 0)$.

Given these five properties, we can now answer our question about continuity at the boundaries. $F$ is continuous at any point on the boundary that serves as the termination point of a classical characteristic. On the other hand, if a point on the boundary is never a termination point for any classical characteristic, it is generally a point of discontinuity for $F$ and therefore corresponds to a concentration of probability mass.

*The state space and Lyapunov functions.* The state space $\mathcal{F}$ of our cumulative distribution functions is a subspace of $L_\infty[0,1]$, the Banach space of bounded real-valued functions on the unit interval $[0,1]$, with norm $||F||_\infty = \sup\{F(x) : x \in [0,1]\}$. Specifically, $\mathcal{F} = \{F \in L_\infty[0,1] : F(0) = 0, F(1) = 1, F$ is non-decreasing and, except possibly at $x = 0$, right-continuous$\}$. Such functions are easily seen to be differentiable almost everywhere, the exceptional points being jump discontinuities.

A sufficient condition for asymptotic convergence of a solution $F(\cdot,t)$ of the Cauchy problem (9) to a particular distribution function $F^* \in \mathcal{F}$ is the existence of a Lyapunov function, that is, a continuous function $V : \mathcal{F} \to \mathcal{R}$ such that (a) the function $t \mapsto V(F(x,t))$ is decreasing for all $x \in [0,1]$ and is strictly decreasing whenever $F(x,t) \neq F^*$, and (b) $V$ attains a global minimum at $F^*$. See Robinson (2001, p274ff) for a considerably more powerful result.

## 7.1 Proofs of Propositions

**Proof of Proposition 1.** To construct $F^*$, first note that the diminishing marginal utility property of $u$ in (2) implies that its derivative $u'$ has a smooth strictly decreasing inverse function $w: [u'(1), u'(0)) \to (0,1]$. If the parameter $c$ in (2) is zero, set $x_o = 0$. If $c > 0$ let $x_o = w(b/c)$ if $b/c \geq u'(1)$; otherwise $x_o = 1$. Define $x_1$ the same way as $x_o$ with $a$ replacing $b$. Since $b > a$ by hypothesis and since $w$ is decreasing, we know that $x_o \leq x_1$, and indeed $x_o < x_1$ when $x_o < 1$.

Recall the function $\hat{F}(x) = \frac{b}{b-a} - \frac{c}{b-a}u'(x)$. Let $F^*$ be given by $F^*(x) = 0$ for $x \leq x_o$, $F^*(x) = \hat{F}(x)$ for $x \in (x_o, x_1)$, and $F^*(x) = 1$ for $x \geq x_1$. By construction, $F^*$ satisfies parts 2 and 3 of the proposition. Moreover, for an arbitrary distribution $G(x)$, inspection of the expressions for $\phi_x$ and $F^*$ reveals:

*Property P.* The gradient $\phi_x(x,G)$ is positive when $G(x) > F^*(x)$, is negative when $G(x) < F^*(x)$, and is zero when $G(x) = F^*(x) = \hat{F}(x)$.

To establish part 1, let $G \in \mathcal{F}$ be the distribution function for a Nash equilibrium of the location game, and suppose that $G(x) > F^*(x)$ for some $x \in [0,1]$. Then by Property P the gradient is positive at $x$, so players at that location are not best responding, contradicting the Nash hypothesis. Similarly $G(x) < F^*(x)$ implies that the gradient is negative at $x$, also contrary to the Nash hypothesis. Hence $G = F^*$ after all.

As for part 4, assume that $c$ is not so large as to preclude conspicuous consumption in equilibrium, i.e., assume $x_o < 1$. Then (as noted earlier) $x_o < x_1$ and the equilibrium utility gradient is zero on the interval $[x_o, x_1]$ by Property P. Hence everyone achieves the same equilibrium utility, which by equation (2) can be written as $\phi(x_1, F^*) = cu(x_1) + b(1 - x_1 - \alpha_{F^*}) + (b-a)\beta_{F^*} = cu(x_1) - a\beta_{F^*}$, where $\beta_{F^*} = \int_0^{x_1} F^*(y)dy = \alpha_{F^*} + x_1 - 1 > 0$. At the clumped (Heaviside) distribution $G(x) = H(x - x_1)$, everyone achieves utility $\phi(x_1, G) = cu(x_1) + b(1 - x_1 - \alpha_G) + (b-a)0 = cu(x_1)$. This last expression exceeds the equilibrium utility by $a\beta_{F^*}$, and it follows that $F^*$ is inefficient iff $a > 0$. ■

**Proof of Proposition 2.** For an arbitrary distribution $F(x)$, recall that $S_F = \{x \in [0,1] : cu'(x) - b > (a-b)F(x)\}$ and that $\widetilde{x}_F = \sup S_F$ if $S_F$ is nonempty, and otherwise $\widetilde{x}_F = 0$. (If $F$ is continuous at $\widetilde{x}$ then it is the unique solution to the equation $cu'(x) - b = (a-b)F(x)$.) Recall from the proof of the previous proposition that typically $x_o = w(b/c)$ and $x_1 = w(a/c)$. Direct computation shows that $\widetilde{x}_F = x_o$ for $F(x) = H(x-1)$ and that $\widetilde{x}_F = x_1$ for $F(x) = H(x-0)$. Intermediate distributions produce intermediate values of $\widetilde{x}$.

Suppose $F(x)$ is the cumulative distribution function for a Nash equilibrium. The hypothesis $a > b$ implies that the gradient $\phi_x(x, F) = cu'(x) - b + (b-a)F(x)$ is decreasing in $x$, and strictly so on points of continuity of $F$. But by definition of Nash equilibrium, $\phi$ is constant (and maximal) on the support of its distribution $F$. Therefore $F$ can't have any points of continuity, nor indeed more than one point of support. Hence $F$ must be of the form $H(x - x^*)$. If $x^* < x_o$ then the gradient is positive at at that point, so a consumer deviating to a higher $x$ gets a higher payoff, contradicting the Nash hypothesis. Likewise, $x^* > x_1$ allows a higher payoff at a lower $x$. Part 1 follows.

For part 2, one verifies for $F(x) = H(x - x^*)$ with $x^* \in [x_o, x_1]$ that $\widetilde{x}_F = x^*$. It follows that nobody can profitably deviate from $x^*$ and that $F$ is a Nash distribution function.

For part 3, one verifies directly that the utility of a consumer in a clumped distribution $H(x - x^*)$ is an increasing function of $x^*$, hence is maximized at $x^* = 1$ where there is no conspicuous consumption. ■

**Proof of Proposition 3.** As noted in the previous section of the Appendix, the results of Bardos et al (1979) establish a unique (weak, entropy) solution $F(x,t)$ to the Cauchy problem (9), which is the appropriate interpretation of the master equation. The proof of part 1 will be complete after verifying that $F(x,t)$ is indeed a cumulative distribution function, i.e., that $F \in \mathcal{F}$. Given the boundary conditions at $x = 0$ and $x = 1$ and the convention on right- continuity already adopted, the only remaining task is to verify that $F(x,t)$ is monotonically increasing in $x$ for every $t > 0$.

The key is property 5) of characteristic curves. Let $0 < x_0 < x_1 < 1$ and $t_0 > 0$. First consider the case where $(x_0, t_0)$ and $(x_1, t_0)$ are on classical characteristics that do not intersect. Since $F$ monotonically increases from 0 to 1 on the initial condition, $t = 0$, while $F = 0$ at $x = 0$ and $F = 1$ at $x = 1$, we see from the fact that $F$ is constant on the characteristics that $F(x_0, t_0) \leq F(x_1, t_0)$. On the other hand, by property 5), if the characteristics do intersect, they must belong to the same rarefaction fan. Since these characteristics only intersect at $(x_D, 0)$, the point of origination of the rarefaction fan, we must have that $\frac{d\chi_0(t)}{dt} \leq \frac{d\chi_1(t)}{dt}$ at $(x_D, 0)$ where $\chi_0$ is the classical characteristic emanating backwards from $(x_0, t_0)$ and $\chi_1$ is the classical characteristic emanating backwards from $(x_1, t_0)$. The defining ODEs for characteristics, (12) and (13), give us $(cu'(x_D) - b) + (b-a)F(x_0, t_0) \leq (cu'(x_D) - b) + (b-a)F(x_1, t_0)$. Since $(b - a) > 0$ by hypothesis, we conclude that $F(x_0, t_0) \leq F(x_1, t_0)$, completing part 1.

For parts 2 and 3, recall the construction of $F^*(x)$ and Property P: the gradient is positive when $F(x,t) > F^*(x)$, is negative when $F(x,t) < F^*(x)$, and is zero when $F(x,t) = F^*(x) = \hat{F}(x)$. Intuitively, Property P ensures stability. Consumers adjust $x$ upward when the gradient is positive, thus reducing the fraction $F(x,t)$ with smaller choices of $x$, and they adjust $x$ downward when the gradient is negative, thus increasing $F$ at the relevant point. This suggests that the distribution converges monotonically pointwise to $F^*$.

To formalize the idea, we verify that $V = \int_0^1 (F(x,t) - F^*(x))^2 dx$ is a Lyapunov function. Clearly Lyapunov property (b) holds by construction. As for (a), we have $dV/dt = 2\int_0^1 (F(x,t) - F^*(x))F_t(x,t)dx = -2\int_0^1 (F(x,t) - F^*(x))\phi_x(x,t)F_x(x,t)dx < 0$. The second equality uses the master equation (5) at points of continuity $x$, for which the density $F_x$ is finite, in the current distribution $F(\cdot, t)$. In the neighborhood of jump discontinuities $x$ in $F(\cdot, t)$, one simply uses the Stieltjes integral, replacing $F_x(x,t)dx$ by $F(dx,t)$ in the last expression. In either case, Property P ensures that $(F(x,t) - F^*(x))\phi_x(x,t)$ is positive at every point $x$ that $F(x,t) \neq F^*(x)$. Further, if there is a value of $x$ where $F(x,t) - F^*(x) \neq 0$ but $F_x(x,t) = 0$, then there also must be an interval of positive $F$-measure contained in [0,1] where $F(x,t) - F^*(x) \neq 0$. Hence (a) holds and parts 2 and 3 follow. ■

**Proof of Proposition 4.** *Lemma.* Let $\text{sgn}[z] = -1, 0$, or 1 indicate whether $z$ is negative, zero or positive. For all $x \in [0,1]$ and all $t \geq 0$, we have $\text{sgn}[F(x,t) - \frac{cu'(x)-b}{a-b}] = \text{sgn}[x - \widetilde{x}]$.

*Proof of Lemma.* Fix $c > 0$ and assume for the moment that $F(x,t)$ is a smooth function of $x$ for all $t \geq 0$. Let $x = \widetilde{x} \in (0,1)$ solve $\frac{cu'(x)-b}{a-b} = F_o(x)$. The solution is unique and interior because the general gradient expression (4) is continuous, strictly increasing, negative at $x = 0$ and positive at $x = 1$. Hence by construction the Lemma holds at time $t = 0$. Moreover, the gradient by construction is zero at $(x,t) = (\widetilde{x}, 0)$, so no mass moves past the point $x = \widetilde{x}$ at $t = 0$ in either direction. The same flow restriction holds at later times by the same argument. Thus $F(\widetilde{x}, t) = F_o(\widetilde{x})$ holds in $t \geq 0$, and the Lemma holds for all $t \geq 0$.

In the case in which $F(x,t)$ is discontinuous in $x$, define $\widetilde{x}$ as in the Proposition. The previous argument is unchanged by discontinuities away from $\widetilde{x}$, so suppose that there is a discontinuity at $x = \widetilde{x}$ for some $t \geq 0$. For small $\epsilon > 0$ the same argument shows that $F(\widetilde{x} - \epsilon, t)$ is decreasing and that $F(\widetilde{x} + \epsilon, t)$ is increasing in $t$ at all points of continuity. Thus the expressions don't change sign and the conclusion still holds: the point $x = \widetilde{x}$ absorbs mass from both directions. Finally, for the case $c = 0$, it is straightforward to verify that the Lemma holds with $\widetilde{x} = 0$.■

*Proof of Proposition.* Part 1 is exactly the same as in the previous proposition, except that we no longer have the hypothesis that $b - a > 0$. However, property 4) of characteristic curves tells us that in the present case, $b - a < 0$, there can be no rarefaction fan, so the desired monotonicity comes directly from property 5).

For parts 2 and 3, we construct the Lyapunov function $V(t) = \int_0^1 (x - \widetilde{x})^2 F(dx, t)$. The integrand is zero at $x = \widetilde{x}$ and where $F(x,t)$ is locally constant. Elsewhere the integrand is positive, and thus Lyapunov condition (b) is verified. To verify (a), integrate by parts to obtain $V(t) = (1 - \widetilde{x})^2 - 2\int_0^1 (x - \widetilde{x}) F(x,t) dx$. Hence the time derivative exists and is equal to $\dot{V} = -2\int_0^1 (x - \widetilde{x}) F_t(x,t) dx$. Using (9), we have $\dot{V} = -2\int_0^1 (x - \widetilde{x})[(a-b)F(x,t) + b - cu'(x)]F(dx,t)$. The Lemma now tells us that $\dot{V}$ is negative except at $F(x,t) = F^*$, and (a) follows. Together with the results from Proposition 2, this establishes parts 2 and 3. ■

**Proof of Proposition 5.** The functions are defined as follows.

$$T(q) = \left[\frac{\overline{\gamma}_z}{f(q)} - \frac{\overline{\gamma}_{zz}}{f''(q)}\right],$$

$$\hat{X}(q) = \left[\overline{\gamma} - \frac{\overline{\gamma}_z F(q)}{f(q)} + \frac{\overline{\gamma}\overline{\gamma}_{zz} F(q)}{f''(q)}\right],$$

where, defining $Q = \ln \frac{qf(q)}{qf(q)-F(q)}$,

$$\overline{\gamma} = \frac{Q}{F(q)};$$

$$\overline{\gamma}_z = \frac{1}{qF(q)} - \frac{Qf(q)}{F^2(q)};$$

$$\overline{\gamma}_{zz} = \frac{2Qf(q)}{F^3(q)} - \frac{2qf(q)+F(q)}{q^2F^2(q)}.$$

The next section derives an implicit solution (19) to the given initial value problem. The solution can be written as $0 = \alpha_1 \equiv tF(z) - z + x - c\gamma$, where

$$\gamma(c, z, x) \equiv \frac{1}{F(z)} \ln \frac{|zF(z) - c|}{|xF(z) - c|},$$

and $z$ is the auxiliary variable defined by $F(z) = F(x, t)$.[10] Shocks initiate where the implicit solution becomes singular, which we may express by differentiating $\alpha_1$ with respect to $z$ to obtain $0 = \alpha_2 \equiv 1 - tf(z) + c\gamma_z$. We also seek the earliest time where a singularity occurs, i.e., a minimal positive value of $t$ in the equation $0 = \alpha_2$. The associated first order condition (obtained by solving the equation for $t$, differentiating and simplifying) is $0 = \alpha_3 \equiv cf(z)\gamma_{zz} - (c\gamma_z + 1)f'(z)$.

For fixed $c \geq 0$, let $\Phi(c, \cdot, \cdot, \cdot) : [0, 1]^2 \times \mathbf{R}_+ \to \mathbf{R}^3$ map the point $(z, x, t)$ to $(\alpha_1, \alpha_2, \alpha_3)$. The differentiability hypothesis in the proposition assures that $\Phi(c, \cdot, \cdot, \cdot)$ is twice continuously differentiable. Consider first the case $c = 0$. Here $0 = \alpha_3$ implies $f'(z) = 0$, so the shock first appears on the characteristic curve associated with the interior maximum $z = q$. From $0 = \alpha_2$ we infer that the shock initiates at time $t^*(0) = 1/f(q)$, and from $0 = \alpha_1$ we infer that the initial shock location is $x^*(0) = z - t^*F(z) = q - F(q)/f(q)$. It is clear that $0 < x^*(0) < q < 1$. We have $x^*(0) > 0$ because, at the global maximum $q > 0$ of $f(q)$, $F(q) = \int_0^q f(y)dy < qf(q)$. Thus we have an interior shock emerging in finite time for $c = 0$.

[10]We apologize for the notation. Usually the initial condition is denoted $F_o$ and the solution is denoted $F(\cdot, t)$. Here we drop the subscript on the initial condition to streamline slightly the rather cumbersome formulas.

For small positive $c$, we apply the implicit function theorem to $\Phi(c, \cdot, \cdot, \cdot)$. The key condition (see e.g. Spivak, 1965) is that the Jacobian determinant $|\mathbf{J}(\alpha_1, \alpha_2, \alpha_3; z, x, t)|$is not zero when evaluated at the point $c = 0$, $z = q$, $x = q - F(q)/f(q)$, and $t = 1/f(q)$. Explicitly,

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial \alpha_1}{\partial z} & \frac{\partial \alpha_1}{\partial x} & \frac{\partial \alpha_1}{\partial t} \\ \frac{\partial \alpha_2}{\partial z} & \frac{\partial \alpha_2}{\partial x} & \frac{\partial \alpha_2}{\partial t} \\ \frac{\partial \alpha_3}{\partial z} & \frac{\partial \alpha_3}{\partial x} & \frac{\partial \alpha_3}{\partial t} \end{vmatrix} = \begin{vmatrix} tf(q) - 1 & 1 & F(q) \\ tf'(q) & 0 & -f(q) \\ f''(q) & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & F(q) \\ 0 & 0 & -f(q) \\ f''(q) & 0 & 0 \end{vmatrix} = -f(q)f''(q).$$

At a regular maximum $q$ of $f$ we have $f''(q) < 0 = f'(q) < f(q)$. Hence the Jacobian is strictly positive and the desired implicit functions exist. They have derivatives (evaluated at the same point) given by

$$\begin{pmatrix} z^{*\prime}(0) \\ x^{*\prime}(0) \\ t^{*\prime}(0) \end{pmatrix} = -\mathbf{J}^{-1} \begin{pmatrix} \frac{\partial \alpha_1}{\partial c} \\ \frac{\partial \alpha_2}{\partial c} \\ \frac{\partial \alpha_3}{\partial c} \end{pmatrix} = - \begin{pmatrix} 0 & 0 & \frac{1}{f''(q)} \\ 1 & \frac{F(q)}{f(q)} & \frac{-F(q)}{f(q)f''(q)} \\ 0 & \frac{-1}{f(q)} & \frac{1}{f(q)f''(q)} \end{pmatrix} \begin{pmatrix} -\gamma \\ \gamma_z \\ f(q)\gamma_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{f(q)\gamma_{zz}}{f''(q)} \\ \gamma - \frac{\gamma_z F(q)}{f(q)} + \frac{\gamma_{zz} F(q)}{f''(q)} \\ \frac{\gamma_z}{f(q)} - \frac{\gamma_{zz}}{f''(q)} \end{pmatrix}.$$

The values of $\gamma$ and its derivatives are readily calculated at $c = 0$. Using the notation $Q = \ln|z| \,/\, |x| = \ln \frac{qf(q)}{qf(q) - F(q)}$, we find that at the relevant point [$z = q$; $x = q - F(q)/f(q)$; $t = 1/f(q)$] we have

$$\gamma = \frac{Q}{F(q)},$$

$$\gamma_z = \frac{1}{qF(q)} - \frac{Qf(q)}{F^2(q)},$$

and

$$\gamma_{zz} = \frac{2Qf(q)}{F^3(q)} - \frac{2qf(q) + F(q)}{q^2F^2(q)}.$$

The expressions in the Proposition now follow from the first order Taylor expansion at $c = 0$. They are valid as long as the shock position $x^*(c)$ remains above the zero $\widetilde{x}(c)$ of the gradient $\phi_x$ corresponding to the condition $x/c = F(x)$. Clearly $\widetilde{x}(0) = 0$, and $\widetilde{x}(c)$ is continuous in $c$ because $F$ has a density, so the condition $x^*(c) > \widetilde{x}(c)$ holds for sufficiently small $c$. ■

## 7.2 Specifications, Computations and Derivations

*Utility functions.* An alternative specification to (2) is that the utility from conspicuous consumption is simply its rank, e.g., Frank (1985) and Hopkins and Kornienko (2004). That is, in our notation, $r(x, F) = F(1-x)$ or perhaps $r(x, F) = 1 - F(x)$ instead of $r(x, F) = \int_0^x (y-x)dF(y)$ as in pure envy or or $r(x, F) = \int_x^1 (y-x)dF(y)$ as in pure pride. For our purposes, such a direct specification has an obvious problem: a small difference in $x$ can produce a very large difference in $r$ when the distribution is concentrated. It may be less obvious that the specification also has the opposite problem, that a relatively large difference in $x$ can produce an arbitrarily small difference in $r$. To illustrate, consider a two-class society with mass 1/2 at $x_L = 0.9$ and mass 1/2 at $x_U = 0.1$, with conspicuous consumption bundles $1 - x_L = 0.1 < 0.9 = 1 - x_U$ that differ by $D = 0.8$. Let $\epsilon = 0.01$ and consider the median consumer in two scenarios: (a) his conspicuous consumption $1-x = 1-x_L+\epsilon = 0.11$ is almost the same as those below him but far less than those above, and (b) the reverse case where his conspicuous consumption $1 - x = 1 - x_U - \epsilon = 0.89$ is almost the same as all those above him but far more than those below. The direct specification makes the median consumer indifferent between (a) and (b), with $r = 0.5$ in either case. Thus a large relative and absolute difference in $x$ produces no difference in $r$. By contrast, in our chosen specification the difference in $r$ is quite large, approximately $D/2$ in the pure cases. The calculations in the pure envy case are, for scenario (a), $r = -\int_0^{0.89}(y-.89)dF(y) = 0.5(.10-.89) = -0.395$, while for scenario (b) $r = -\int_0^{0.11}(y - .11)dF(y) = 0.5(.10 - .11) = -0.005$, an improvement of 0.39.

Such examples persuade us that our chosen specification is superior for our purposes.[11] In a setting where conspicuous consumption is modeled explicitly as a noisy signal of true rank, the argument against the direct specification is even stronger. In that setting, large differences in true rank clearly could not be signalled by arbitrarily small differences in consumption shares.

Our specification reduces to “keeping up with the Joneses” in the special case that $a = b > 0$ and $F$ is a Heaviside function. In that case, the utility landscape depends only on the

[11] Hopkins and Kornienko (2004) modify the direct specification by introducing a parameter $\gamma$ that decreases $r(x)$ when $x$ is a mass point. However, their modified $r$ (or $S$ in their notation) remains discontinuous as one passes from a concentrated density to a mass point.

population mean $\bar{x}$.

*Pride computation.* Recall that section 4.2 focuses on the example $\phi(x,F) = c\ln x + \int_x^1 (y - x)dF(y) = c\ln x + 1 - x - \int_x^1 F(y)dy$ and that the equilibrium distribution for $x \in [0,1)$ is $F(x) = [1 - c/x]_0^1$, where the notation $[z]_0^1 \equiv \min\{1, \max\{0, z\}\}$ denotes truncation. Thus for $x \geq c$ we have $\int_x^1 F(y)dy = \int_x^1 (1 - c/y)dy = 1 - x + c\ln x$, while for $x < c$ the expression becomes $\int_c^1 (1-c/y)dy = 1-c+c\ln c$. We conclude that $\phi(x,F) = c\ln x + 1 - x - (1-c+c\ln c)$ for $x \in (0,c)$ and $\phi(x,F) = 0$ for $x \in [c,1]$.

To verify that utility is negative when $x < c$, write $\phi(x,F) = c(\ln x/c + 1 - x/c)$ and Taylor expand $\ln(1-h)$ with $h \equiv 1 - x/c > 0$ to obtain $\phi(x,F) = -c(h^2 + h^3/3 + h^4/4 + ...) < 0$.

*Derivation of analytic solution to the master equation.* We consider the case of pure envy with log ordinary utility. The master equation (5) becomes

$$F_t - [F - (c/x)]F_x = 0, \tag{14}$$

while the ordinary differential equations (12, 13) defining the characteristic path $x = \chi(t)$ and the behavior of $F$ on the characteristic path become

$$\frac{d\chi(t)}{dt} = \frac{c}{\chi(t)} - F(\chi(t), t) \tag{15}$$

$$\frac{dF(\chi(t), t)}{dt} = 0. \tag{16}$$

Recall that (16), along with properties 3) and 4) of characteristic curves, tell us that $F$ is constant along any characteristic, including at the initial condition.

Because $F$ is constant, it is useful to define an auxiliary variable $z = z(\chi(t), t)$ implicitly given by $F(\chi(t), t) = F_0(z)$ so each characteristic is labeled by its corresponding value of $z = \chi(0)$. Use the streamlined notation $x = \chi(t)$, separate variables and substitute $F_o(z)$ for $F$, to obtain the expression $dt = xdx/(c - xF_o(z))$. Because $z$ is fixed along any characteristic, this expression can be integrated directly[12] using the textbook formula

$$\frac{x}{1-ax} = \frac{1}{a^2}\frac{d}{dx}[1 - ax - \ln(1-ax)].$$

We obtain

$$t + t_0(z) = \frac{1 - xF(z)/c - \ln|1 - xF(z)/c|}{F^2(z)/c}. \tag{17}$$

[12] We thank Joel Yellin for this idea.

In (17), the integration constant $t_0(z)$ is constant along each characteristic but varies across characteristics. By the definition of $z$, $x = z$ at $t = 0$. Hence

$$t_0(z) = \frac{1 - zF(z)/c - \ln|1 - zF(z)/c|}{F^2(z)/c}, \tag{18}$$

and we may subtract (18) from (17) to obtain the implicit solution

$$t = \frac{z - x}{F(z)} + \frac{c}{F^2(z)} \ln\left(\frac{|c - zF(z)|}{|c - xF(z)|}\right) \tag{19}$$

of the master equation.

*Shock wave computation.* We continue the previous example and now derive the behavior over time of the shock position and magnitude, given an initial probability density $f_o(x)$ and corresponding cumulative distribution $F_o(x)$. To obtain an exact analytic solution, we eventually will specialize to the $c = 0$ limit; Proposition 5 shows how to extend the solution to small positive $c$. The exposition is adapted from Whitham (1974, p.42-46); see also p.96-112 for an analysis of Burgers' equation (corresponding to our $c = 0$ pure envy case) via the Cole-Hopf transformation. The derivation of the shock initiation time and magnitude, given initial symmetric beta density, is new and due to Joel Yellin.

In general, three conditions determine the shock position $s(t)$ and the leading and trailing values $z_L$,$z_R$ that mark the left and right edges of the shock in terms of the auxiliary variable $z$. The first two conditions apply the solution (19) of the underlying dynamic equation to the leading and trailing edges of the shock, viz.

$$t = \frac{z_L - s}{F(z_L)} + \frac{c}{F^2(z_L)} \ln\left(\frac{c - z_L F(z_L)}{c - sF(z_L)}\right); \tag{20}$$

$$t = \frac{z_R - s}{F(z_R)} + \frac{c}{F^2(z_R)} \ln\left(\frac{c - z_R F(z_R)}{c - sF(z_R)}\right), \tag{21}$$

The third condition is the Rankine-Hugoniot condition. Here the source term is $h = c/x - F^2$, so (11) simplifies to

$$\frac{ds}{dt} = -\frac{1}{2}[F(z_L) + F(z_R)] + \frac{c}{s}. \tag{22}$$

For $c = 0$, (20) and (21) can be combined and expressed in the symmetric forms

$$s = \frac{1}{2}\left[z_L + z_R\right] - \frac{t}{2}[F(z_L) + F(z_R)], \tag{23}$$

$$t = \frac{z_R - z_L}{F(z_R) - F(z_L)}. \tag{24}$$

Furthermore, on differentiating (23) with respect to $t$, equating the result with the $c = 0$ form of (22), and substituting for $t$ from (24), we obtain

$$[z_R - z_L]\,[F'(z_L)\dot{z}_L + F'(z_R)\dot{z}_R] = [F(z_R) - F(z_L)]\,[\dot{z}_L + \dot{z}_R]\,, \tag{25}$$

which integrates to the "equal area condition"

$$\frac{1}{2}[F(z_L) + F(z_R)]\,[z_R - z_L] = \int_{z_L}^{z_R} F(z)dz. \tag{26}$$

Equation (26) states that the shock cuts off equal areas from the breaking wavefront, preserving population mass. This interpretation becomes transparent if we rewrite (26) in the form

$$\int_{z_L}^{\widehat{z}} \left\{\frac{1}{2}[F(z_R) - F(z_L)] - F(z) + F(z_L)\right\} dz = \int_{\widehat{z}}^{z_R} \left\{F(z) - F(z_L) - \frac{1}{2}[F(z_L) - F(z_R)]\right\} dz,$$

which explicitly equates the areas swept out by the right and left "lobes" of the shock. Note that the crossover point of the shock, $z = \widehat{z}$, is determined by $F(\widehat{z}) = \frac{1}{2}[F(z_L) + F(z_R)]$.

To determine the shock initiation time $t^*$, it is helpful to write (24) in terms of the density $f(z)$. Introduce $\Delta = [z_R - z_L]/2$, $\overline{z} = [z_R + z_L]/2$, so that $\Delta(t^*) = 0$ when a shock initiates. Then (24) becomes

$$\frac{1}{2\Delta}\int_{-\Delta}^{\Delta} f(z + \overline{z})dz = \frac{1}{t}, \tag{27}$$

and in the limit $\Delta \to 0$ we have

$$\frac{1}{t^*} = f(\overline{z}). \tag{28}$$

Certain basic relations follow from the assumed unimodality and symmetry about $z = 1/2$ of the density $f(z)$. Symmetry $f(z + 1/2) = f(-z + 1/2)$ gives $\overline{z} = 1/2$,

$$\int_0^{1/2-\Delta} f(z)dz = \int_{1/2+\Delta}^{1} f(z)dz,$$

and therefore

$$F(z_L) + F(z_R) = F(1/2 - \Delta) + F(1/2 + \Delta) = 1. \tag{29}$$

From $\overline{z} = 1/2$, (23) and (29), the position of shock at time $t$ is

$$s(t) = \frac{1}{2}(1 - t),\ t > t^*, \tag{30}$$

where from (28), the shock initiation time satisfies

$$\frac{1}{t^*} = f(1/2). \tag{31}$$

In the symmetric $c = 0$ case, the shock therefore reaches position $s = x = 0$ at $t = 1$.

The shock initiation time and magnitude now follow for the special case described in the text. This solution is one instance from an infinite class of analytic, $c = 0$ shock solutions for the conservation law (9), for the family of initial unimodal beta densities symmetric about $z = 1/2$,

$$f_a(z) = \frac{z^{a-1}(1-z)^{a-1}}{B(a,a)}, \ a > 1, \tag{32}$$

with corresponding distributions $F_a(z)$, where the beta function $B(a,a) = [(a-1)!]^2/(2a-1)!$.

Equations (31, 32) give the shock initiation time

$$t^* = \frac{2^{1-2a}}{B(a,a)}. \tag{33}$$

From (24), the shock magnitude is given implicitly by

$$2F_a(1/2 + \Delta) - 1 = \frac{2\Delta}{t}. \tag{34}$$

For the beta density $a = 2$ considered in the text, $F_2(z) = z^2(3 - 2z)$, and (34) reduces to a quadratic equation in $\Delta$,

$$\frac{1}{t} = \frac{3}{2} - 2\Delta^2. \tag{35}$$

From (35) or (31), the shock initiates at $t^* = 2/3$. Equation (30) gives the initial shock position $x^* = 1/6$, as stated in the text. The shock magnitude thereafter is given by (35).