

Evolving Landscapes for Population Games

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Feb. 17, 1997[‡]

Abstract

We consider population games where the possible actions of each player are labeled by a real number that ranges over a finite interval. The adjustment dynamics of such games can be visualized as motion over a “landscape” - the surface defined by a payoff or fitness function. A leading example is gradient dynamics, in which the speed with which a player changes action is proportional to the gradient (or slope) of the landscape at his current action. We show that gradient dynamics arise from individual optimizations, given the costs of changing actions. We also show that the time behavior of the action distribution in gradient dynamics is described by a class of nonlinear integro-partial differential equations with deviating spatial arguments. We solve these equations analytically for some interesting choices of payoff functions. Cases are exhibited in which the distribution of actions develops compression and rarefaction shock waves. The results of numerical simulations are presented. We characterize the limiting probability distributions of classes of population games, and find sufficient conditions for convergence to pure Nash equilibrium and for convergence to distributions with full support. Applications are suggested in economics and population biology.

1 Introduction

Virtually all the existing work on the adjustment dynamics of population games deals with choice from a finite *unordered* set A of alternative strategies or actions.¹ We propose a

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[‡]We thank Jamie Wylde for research assistance. We thank Peter Abrams, Ilan Eshel, Richard Montgomery, Tom Nagylaki, Robert Rosenthal, and Carl Simon for helpful comments. This paper benefited from discussion at the Stanford Biological Sciences Seminar, the Non-linear Dynamics Seminar at University of California, Santa Cruz, and the Stony Brook Game Theory Workshop, Summer 1996.

¹The literature on the dynamics of population games has grown rapidly in the last fifteen years, first in biology [MS82], [EA83], [TJ78], [HS88], and more recently in economics and game theory [Bin87], [FK88];

theory of evolution processes for strategically interacting populations when the action set A has an ordering or topology. The action space population distribution plays the role of state variable, and we study its evolution over time. We assume that individuals adjust with finite velocity, changing actions to nearby actions only. Given this *local adjustment path* (LAP) restriction, the rate of change of actions is governed by a velocity field on the action space. The existence of a velocity field allows us to write down a master nonlinear integro-partial differential equation for the population distribution. We discuss the properties of the master equation below, giving analytic and numerical solutions for illustrative special cases.

The study of population games under LAP dynamics raises basic questions: What is the relation between the static equilibria of a population game — Nash equilibria (NE) and evolutionarily stable strategies (ESS) — and states that are invariant under adjustment dynamics? Under what conditions does an evolving state converge over time to an invariant distribution, rather than fluctuating chaotically or cycling? Under what conditions does a state converge on, or move away from, extreme behavior at the boundaries of action space? More generally, under what conditions does *clumping*, in which subgroups or an entire population choose the same action, develop or disperse?

We address these questions here, exploiting a geometric representation of population games on the unit interval $A = [0, 1]$. We interpret such games as taking place on *landscapes* with peaks and valleys that pose dynamic obstacles and coevolve with the distribution of actions.² At any time, the landscape has a shape determined by the rules of the game and by the contemporary state of the population. Hence the landscape is integral to a feedback system. The shape of the current landscape induces players to adjust their actions and hence alters the action distribution or state. But altering the current state induces deformations in the landscape and so induces further changes in both the state variable and the landscape.

That states of population games develop from mutual interactions between players and landscape is analogous to the notion that organisms arise and evolve from the unique mutual interactions of genotype and environment [Lew92]. In the discussion to follow, we explore the interplay between the current state and the current landscape, and their evolution over time.

Special cases of LAP dynamics appear in a number of physical and biological contexts. For example, the laws of fluid dynamics also play out on landscapes and incorporate the LAP restriction of no discontinuous jumps in location. But disturbances in fluids propagate at finite speed, so that behavior at distant points has no immediate local impact.³ In population biology, standard dynamical models are less restrictive. They allow action at a distance in

see [Mai92] and [Cra95] for surveys and [Wei95] and [FL95] for texts.

²The landscape metaphor is due to Sewall Wright, e.g. [Wri49]; see [Kau93] for a recent overview. Landscapes considered in population biology are defined on time-invariant sequence spaces of very high dimension, while the landscapes studied here are time-varying and defined on a low-dimensional action space. Simulations in low-dimensional spaces are also in use in studies of political dynamics [Lom97], [KMP96].

³The laws of fluid dynamics have particular relevance here. It will be shown that the probability mass comprising the action distribution behaves as an inviscid fluid.

trait space, though only via the population average. Evolutionary game theory takes yet another approach. It allows the entire distribution of actions or traits to matter locally, but only as it affects the expectation of an underlying two-player (or simple normal form) payoff function. Our LAP framework synthesizes concepts from these separate literatures. It provides a natural approach to the competitive dynamics of populations, extending the game theoretic approach to allow for more general payoff functions and action distributions, while relaxing the topological restrictions in fluid dynamics and generalizing the dynamical averaging processes used in population biology.

This article is organized as follows. In Section 2 we define population games as an extension of normal form two-player games. We contrast pairwise payoff functions with the distributed payoff functions of general population games. In Section 3 we analyze gradient dynamics on the discrete grid $A = \{0, 1/N, 2/N, \dots, 1\} \subset [0, 1]$ when the distributed payoff function is derived from a pairwise payoff function of a symmetric two-player game. We solve explicitly for the action distribution in a fundamental case. The solution exhibits a moving discontinuity analogous to a shock front in fluid dynamics. The discrete formulation also provides an algorithm for numerical solutions of our models. Section 4 develops the geometry of landscapes as a metaphor for a strategically interacting population. Section 5 presents adjustment processes formally, emphasizing gradient processes and a generalization we call sign-preserving dynamics. We derive the gradient process as the optimal response to quadratic adjustment cost, extending results previously obtained by [Son82]. We solve some salient examples analytically and numerically.

The nature of invariant distributions is explored in Section 6. We generalize the equal expected payoff property of mixed Nash Equilibria in standard games. We also generalize the Bishop-Cannings Theorem [MS82] to continuous action spaces. After adapting standard static and dynamic definitions of equilibrium to our setting, we characterize the invariant distributions of all generalized gradient processes, both clumped and dispersed. In Section 7, we consider generalized gradient processes and find sufficient conditions for the population mean payoff to increase over time, and for asymptotic convergence to an invariant distribution. Section 8 examines transient behavior and gives fully solved examples of compression shock fronts in the probability density of actions. Section 9 explores dispersed behavior, using as illustration the classic centered expansion wave or “fan.” Section 10 describes clumping of actions. We find sufficient conditions for convergence to dispersed and to clumped distributions and point out connections to existing equilibrium results in game theory and population biology.

Section 11 applies our results to several well studied models in biology and economics, including Cournot duopoly, the war of attrition [MS82] and the Edgeworth cycle, the [Son82] model of spatial competition, and continuous biological traits. In these applications, we present a dynamic picture that complements standard static results. The paper concludes in Section 12 with a brief discussion of open questions and possibilities for further applications. Our notation, derivations, and proofs are collected in an Appendix.

2 Population Games

The dynamics of population games are determined by the interplay of two mathematical objects: an action distribution D (or density ρ) and a distributed payoff function ϕ or pairwise payoff function g . We define these terms here for the simplest relevant case: a single large population of players interacting symmetrically. We leave for future work generalizations to games in which there are strategically distinct subpopulations, e.g., buyers and sellers or males and females.

Except when otherwise stated, we take the action set to be the continuous unit interval $A = [0, 1]$. Action distributions are probability measures on the action set. We represent a measure by its cumulative distribution function D or by its density ρ . A cumulative distribution function on $A = [0, 1]$ is a non-decreasing right-continuous function $D : \mathcal{R} \rightarrow \mathcal{R}$ such that $D(x) = 0$ for $x \leq 0$ and $D(x) = 1$ for $x \geq 1$. The associated density $\rho = D_x$, provided the derivative D_x exists everywhere in A .

◆Let \mathcal{D} denote the set of all distributions on $[0, 1]$. The *distributed payoff function* $\phi : [0, 1] \times \mathcal{D} \rightarrow \mathcal{R}$ assigns the payoff $\phi(x, D)$ to any player choosing x , given that D describes the distribution of other players' actions.

We assume the player population is so large that the decisions of a single player have no significant effect on the current action distribution. Hence, we use the same action distribution D in the distributed payoff function for all players.

◆Let $g(x, y)$ be the *pairwise payoff function* for an arbitrary, symmetric two-player normal form game on the square. Then $g(x, y)$ is a measurable real-valued function on $[0, 1] \times [0, 1]$ that represents the payoff for choosing action x when the opponent has chosen action y .

An important class of distributed payoff functions are derivable from pairwise payoff functions as follows. Suppose players in a population game associated with a pairwise payoff function $g(x, y)$ are “randomly matched” in pairs and have the current action distribution D . Then the expected payoff to a player choosing action x is $\phi(x, D) = E_D g := \int_0^1 g(x, y) dD(y)$. If the density ρ exists, we write $\phi(x, D) = \int_0^1 g(x, y) \rho(y) dy$.

3 Discrete Gradient Dynamics

We illustrate population games and introduce our dynamical rules by considering a population game played out in continuous time on an ordered, discrete action space. The dynamics are governed by a finite system that describes the population flow — or the flow of probability mass — among neighboring points on a one-dimensional lattice labeled $i = 0, 1, \dots, n$. Each lattice point represents a possible action. Let the array $((g_{ki}))$ be the pairwise payoff

for choosing action k when the opponent has chosen i . Let $\rho_k = N_k/N$ be the fraction of players choosing action k . Our interest here is in an associated population game played by a large number of individuals N .

In the population game, the distributed payoff to action k is the expectation

$$\phi_k = \sum_i g_{ki} \rho_i \quad (1)$$

of the pairwise payoff. We make the dynamical assumption that probability mass flows only between neighboring points. In particular, we implement the following rule:

MASTER RULE. *All flows from point k to the neighboring points $k \pm 1$ only take place if they are “uphill” in terms of the distributed payoff. That is, mass flows from point k to point $k - 1$ if and only if the first difference $X_k \equiv \phi_{k-1} - \phi_k$ is positive; mass flows from $k - 1$ to k if X_k is negative. When flow does occur, it is proportional to the mass ρ_k at sourcepoint k and to the payoff difference X_k .*

The Master Rule embodies the conservation of probability in the form

$$d/dt (\text{probability density}) = - \text{outgoing flow} + \text{incoming flow}.$$

Formally, the population dynamics are then described by the real time $(n + 1)$ -dimensional discrete evolution system

$$d\rho/dt = \mathbf{M}(\mathbf{X}) \cdot \rho, \quad (2)$$

where the evolution matrix is ⁴

$$\mathbf{M}(\mathbf{X}) = \begin{pmatrix} X_1^- & X_1^+ & 0 & 0 & 0 & 0 & 0 \\ -X_1^- & -X_1^+ + X_2^- & \dots & \dots & \dots & 0 & 0 \\ 0 & -X_2^- & \dots & X_k^+ & 0 & 0 & \dots \\ 0 & 0 & \dots & -X_k^+ + X_{k+1}^- & \dots & 0 & 0 \\ 0 & 0 & 0 & -X_{k+1}^- & \dots & X_{n-1}^+ & 0 \\ 0 & \dots & 0 & \dots & \dots & -X_{n-1}^+ + X_n^- & X_n^+ \\ 0 & 0 & 0 & 0 & 0 & -X_n^- & -X_n^+ \end{pmatrix} \quad (3)$$

Continuum limit. In the limit of a continuous action space, (1) becomes the integral relation $\phi(x, t) = \int_0^1 g(x, y) \rho(y, t) dy$. We restrict discussion here to discrete evolution equations that have measurable terms in the continuous limit. Thus the quantities X_i in (2) are restricted to a monotone sequence, except possibly at isolated points, and one can write the right-hand side of (2) as a double difference. Explicit calculation of the secular determinant $|\mathbf{M} - \lambda \mathbf{I}|$ shows that the tridiagonal matrix (3) has instantaneous negative eigenvalues, hinting at convergence properties we shall discuss below in Section 7. In the continuous limit (2) becomes

$$\rho_t(x, t) = -[\rho(x, t) \phi_x(x, t)]_x. \quad (4)$$

⁴In (3), $x^+ = \max\{0, x\}$, and $x^- = \min\{0, x\}$.

Equation (4) plays a central role in the analysis presented in later sections.

The system (2,3) defines a nonlinear discrete Markov process in which flows switch directions at dynamically determined times. The associated evolution equations in general have no analytic solutions. Throughout this article, we therefore refer to numerical solutions based upon (2,3). Our numerical calculations proceed via the usual Euler discretization, taking an initial $\rho_i(0)$ such that $\sum_i \rho_i(0) = 1, \rho_i(0) \geq 0$, choosing a sufficiently small time step Δ , and iterating (2) by replacing the left-hand side by the forward differences $[\rho_i(t + \Delta) - \rho_i(t)]/\Delta$.

Example: Hierarchical action tower.

To set the stage for what follows, we now give an analytic solution of the discrete system (2,3) for a simple special case. We choose a pairwise payoff that induces a permanent hierarchy of actions $(0, 1, 2, \dots, n)$ such that action k weakly dominates action $k + 1$:

$$\mathbf{g} = - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \dots & \dots & \dots & 0 & 0 \\ \dots & 1 & \dots & 1 & 0 \\ 1 & \dots & \dots & 1 & 1 \end{pmatrix}. \quad (5)$$

The Appendix shows that given (5), equation (2) becomes the nonlinear system

$$\begin{aligned} \dot{\rho}_0 &= \rho_1^2, \\ \dot{\rho}_k &= -\rho_k^2 + \rho_{k+1}^2, \quad k = 1, 2, 3, \dots, n-1, \\ \dot{\rho}_n &= -\rho_n^2. \end{aligned} \quad (6)$$

From (6) we observe that

$$\sum_{k=p}^n \dot{\rho}_k = -\rho_p^2, \quad p \geq 1, \quad (7)$$

so that for $t \rightarrow \infty$ all probability is “clumped” at action 0, with $\rho_0 = 1$.

For the 3×3 case $n = 2$, we solve (6) explicitly in the Appendix. Figure 1 shows the solution for initial conditions $(\rho_0, \rho_1, \rho_2) = (.4, .06, .54)$. As anticipated, all probability ultimately flows to the dominant action 0. In section 8 we solve for the continuous action density $\rho(x, t)$, given the continuous analog of (5), $g(x, y) = \theta(x - y)$.

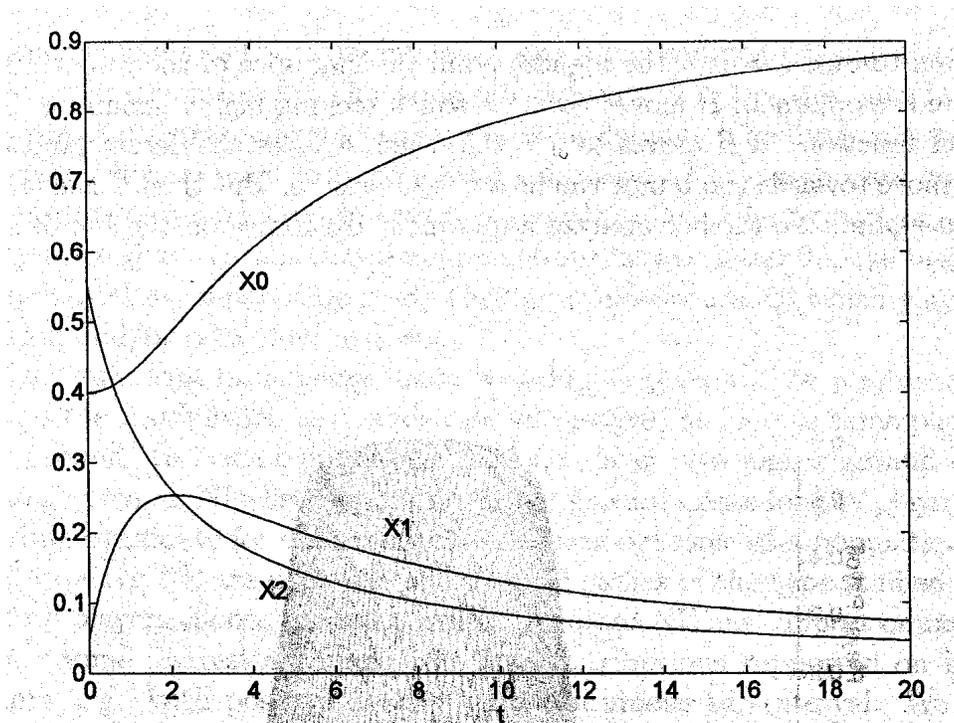


Figure 1: Time behavior of probability densities for three actions (0, 1, 2), given hierarchical 3x3 payoff matrix described in text.

4 Population Games and Landscapes

When the action set is continuous, payoff functions can be visualized as landscapes. The topographic analogy leads to some geometric intuition. For concreteness, let g be a pairwise payoff function defined on the action set $[0, 1]$. The graph of g is a surface over the unit square $[0, 1] \times [0, 1]$. A player \mathcal{A} chooses an action that determines the east-west coordinate x , while the opponent \mathcal{B} chooses an action that determines the north-south coordinate y . The altitude $g(x, y)$ at the point (x, y) is player \mathcal{A} 's payoff. \mathcal{A} then has an incentive to move east or west to reach higher ground.

Figure 2 exhibits the graph of a pairwise payoff function that we shall refer to as the “bundt.”⁵ If the opponent’s action is $y = 0.2$, then player \mathcal{A} has an incentive to shift action toward $x = 0.5$, the highest point on that slice of the bundt. But at the same time the opponent \mathcal{B} moves north or south seeking higher ground in his slice of the payoff function. If \mathcal{B} moves to $y = 0.5$, then \mathcal{A} faces a different landscape slice and will move towards the bundt rim at $x = 0.25$ or 0.75 . But then \mathcal{B} has an incentive to shift y again. We further analyze adjustment

⁵“Bundt” refers to the eponymous Viennese cake that is a favorite dessert of one of the authors. A payoff function with the bundt form can be thought of as arising from a two-person game in which x and y represent the distances from a fixed reference point that corresponds to zero use of a finite resource. The bundt shape appears if the return is maximized at an intermediate level of resource use.

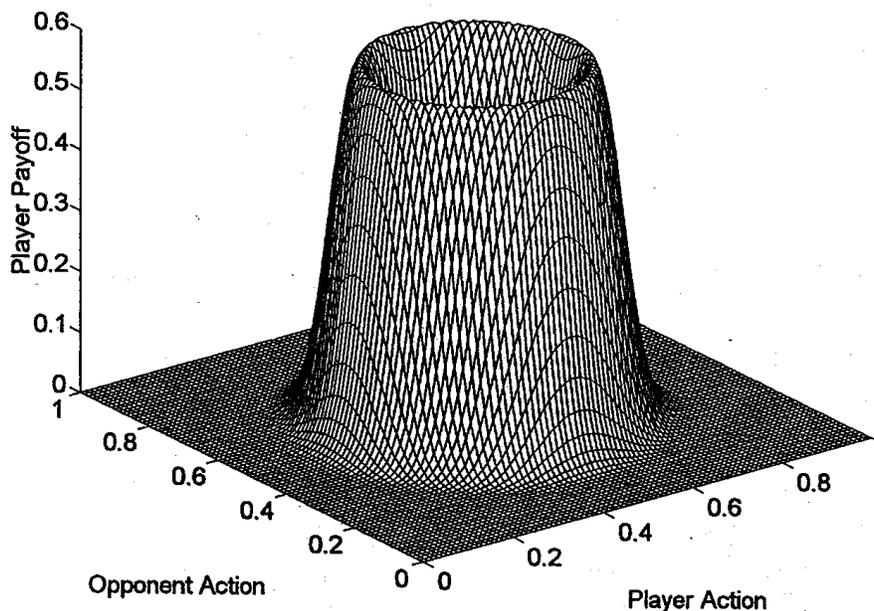


Figure 2: Payoff landscape for pairwise bundt payoff function $g(x, y) = 1024r^4(1 - r)\exp(-512r^4)$, where $r^2 = (x - .5)^2 + (y - .5)^2$.

dynamics on the bundt in Section 5.4 below.

The landscape metaphor extends to a complete population of players. One may visualize such a population as defining a distribution D (or a density ρ if the distribution is smooth) on the north-south axis. Then the relevant landscape (or east-west profile) is the distribution-weighted average of east-west profiles. For example, if the distribution in Figure 2 is concentrated with mass m at the point $y = 0.2$ and mass $(1 - m)$ at the point $y = 0.5$, then the profile has three peaks at $x = 0.25, 0.5$ and 0.75 whose relative heights depend on m .

When the distributed payoff function ϕ cannot be obtained from an underlying pairwise payoff function g , the set of possible landscapes is a family indexed by $D \in \mathcal{D}$. Whether or not ϕ can be obtained from an underlying pairwise payoff function g , the relevant profile evolves because D changes over time as individuals in the population mutually adjust their behaviors.

Adjustment dynamics track the interplay over time between the payoff landscape $\phi(\cdot, D)$ and the distribution D . The landscape deforms as D changes, and D changes as players respond to the current shape of the landscape. The dynamics are sensitive to precisely how players respond to the landscape. Those responses depend on whether players can see the whole east-west profile or just nearby portions, and on whether they are able to jump anywhere

instantaneously or instead move east or west at finite velocity. In this paper we study the finite velocity case in which players have limited vision and/or rate of adjustment.

Given the rule that each player must seek higher ground, the players will keep moving (and the distribution and landscape will evolve) as long as some players can see higher ground. In Nash equilibrium (NE) no player sees higher ground because, by definition, given an NE distribution no higher ground exists for any player. Hence NE is a prime candidate for describing the ultimate outcome of a population game.

Several types of NE are plausible outcomes of games of the type studied here. In a *pure* NE all players choose the same action. A *mixed* NE has support on more than one action. Some interesting mixed NE have distributions supported on infinitely many points, e.g. with positive density on a continuous sub-interval. We refer to such distributions as *dispersed*. We expect dispersed NE for payoff functions that penalize popular strategies, as in the congestion games described below. We check this expectation after formalizing adjustment processes and working through some examples.

5 Landscape Dynamics

Adjustment processes or dynamics describe how behavior in the population changes over time. The most detailed specifications track the stochastic adjustment of a finite population. In this paper we concern ourselves only with the *distribution* of actions and only with *deterministic* processes.⁶

The dynamics of a population game then describe the time path or trajectory, $\mathcal{T} := \{D(\cdot, t) : t \geq 0\}$, of the action distribution in \mathcal{D} as players seek better actions. In deference to Darwin's canon *Natura non facit saltum*, we will assume that change is smooth, and that the rate of change is a function of the current state D and the strategic environment embodied in the distributed payoff function ϕ . Hence a dynamic process in our setting is described by an autonomous evolution equation of the general form

$$D_t(x, t) = \Psi(x, D, \phi). \tag{8}$$

5.1 Replicator and Best Response Dynamics

Dynamic adjustment described by (8) can occur for several reasons. For example, the player population can change because of births and deaths. If so, and if payoffs represent reproductive fitness, then the demographics are driven by natural selection. Ignoring the

⁶See [BS94] for a discussion of the senses in which deterministic processes for infinite populations are limits of stochastic processes for finite populations.

complications of sex and diploidy, the result is *replicator dynamics* [TJ78]. Under replicator dynamics, the growth rate of probability mass at an action x is proportional to the deviation of the distributed payoff $\phi(x, D)$ from the average payoff $\mu_D = \int_0^1 \phi(y, D)\rho(y, t)dy$. Hence, in terms of the density function $\rho(x, t)$ of the current distribution D , replicator dynamics are given by the evolution equation⁷

$$\rho_t(x, t) = \rho(x, t)[\phi(x, D) - \mu_D], \quad (9)$$

where we have absorbed the proportionality constant into the definition of time scale.

More important for many applications is the possibility that the players remain the same but the action distribution changes as individuals change their actions, voluntarily or by coercion. In particular, players may learn from their own experience or from observing other players and adapt their behavior accordingly. Let

$$B(D) := \{x \in [0, 1] : \phi(x, D) \geq \phi(y, D) \forall y \in [0, 1]\} \quad (10)$$

be the best response correspondence, and let $\beta(x, D)$ be the fraction of $B(D)$ at or below x .⁸ Then *inertial best response dynamics* are described by the evolution process

$$D_t(x, t) = c[\beta(x, D(\cdot, t)) - D(x, t)]. \quad (11)$$

The motivating idea for (11) is that in a time interval Δt , a random $O(\Delta t)$ fraction of the population jumps to a best response while other agents' actions remain unchanged. Of course, if the jump fraction is not $O(\Delta t)$ (e.g., if the rate constant c is infinite), the anti-saltation canon is violated and we have “revolution not evolution.”⁹

5.2 LAP Dynamics

In this paper we focus on *local adjustment path* (LAP) dynamics in which Darwin's canon applies to every individual as well as to the population as a whole. Inertial best response dynamics are not LAP. When an individual changes action it is always a jump to a best response. Replicator dynamics are also not LAP. A few individual actions change drastically when individuals are born who bear no relation to the newly deceased. Yet over a sufficiently short time period most players do not change actions, and changes in the overall action

⁷See [Wei95] for a derivation of (9) for finite action spaces that generalizes to the present case of continuous action spaces. See [To95] for an elegant formulation of (9) and an application to the evolution of risk preferences.

⁸When B is a singleton x^* , i.e., the best response is unique, then β is simply 0 for $x < x^*$ and 1 for $x \geq x^*$. Various conventions for β are possible when the best response is not unique. These yield slight variants on equation (11) below.

⁹See [WFCB96] for an application of a discrete time version of (11) to populations of motorists who devote a fraction of their individual resources to accident prevention. See [Wei95] and [FL95] for treatments of the finite action space versions.

distribution are small. In LAP dynamics, by contrast, all players can change their actions in a short period of time, but the individual changes are restricted to be small.

All dynamical population games incorporate a payoff function and current action distribution. LAP dynamics further require a velocity field $v(x, t)$ that describes the adjustment of a player choosing x at time t . Then the general equation (8) takes the form

$$D_t(x, t) = -v(x, t)D_x(x, t). \quad (12)$$

The left hand side of (12) represents the time rate of change of probability mass in $[0, x]$. The right-hand side of (12) represents the rate at which probability mass moves left past a point x , given the velocity field v . Equation (12) therefore states that probability mass is conserved.

It is sometimes more convenient to write the evolution equation (12) entirely in terms of densities. If D is differentiable in x , the expression $D_x(x, t)$ is the density $\rho(x, t)$. Taking the derivative of (12) with respect to x thus yields

$$\rho_t(x, t) = -[v \cdot \rho]_x = -v_x(x, t)\rho(x, t) - v(x, t)\rho_x(x, t), \quad (13)$$

the well-known continuity equation for fluid flow. From another point of view, (13) is the Fokker-Planck-Kolmogorov equation; see e.g. [MM95].

A natural way to connect the velocity field to the payoff function is to say that players move towards higher payoffs, and that they move faster when the payoff rises more steeply. The direct formalization of this idea, called *gradient dynamics*, is the continuous action space version of the discrete dynamics described in Section 2. Gradient dynamics set $v(x, t) = \phi_x(x, D(\cdot, t))$, so that the velocity field in (12) is equal to the payoff gradient. Note that the basic evolution equation is then nonlinear in D (or ρ). In particular, if the distributed payoff function ϕ comes from a pairwise payoff g , then the continuity equation is

$$\rho_t(x, t) = -[Eg_x \cdot \rho]_x = -\int_0^1 [\rho(x, t)g_{xx}(x, y) + \rho_x(x, t)g_x(x, y)]\rho(y, t)dy. \quad (14)$$

A useful generalization of gradient dynamics is *sign preserving*, in that the velocity field at a particular action always has the same sign (but not necessarily the same magnitude) as the payoff gradient. Formally:

◆LAP dynamics are *sign preserving* if there are constants $K_1, K_2 > 0$ such that for all x and t , the velocity $v(x, t)$ is in the closed interval with endpoints $K_1\phi_x(x, \rho(\cdot, t))$ and $K_2\phi_x(x, \rho(\cdot, t))$.

This formal definition prevents v from approaching 0 or ∞ when ϕ_x is bounded away from 0 and bounded above.

The partial differential equations (12) and(13) are undefined when D is not differentiable or v is discontinuous. In all cases we therefore assume probability mass acts as an inviscid fluid

whose flow is characterized by a velocity field. Discontinuities in v can occur at end points because in LAP dynamics we impose the boundary conditions

$$v(0, t) \geq 0 \text{ and } v(1, t) \leq 0 \text{ for all } t \geq 0. \quad (15)$$

The boundary conditions (15) prevent mass from exiting the interval $[0, 1]$. The condition (12) prevents mass from entering $[0, 1]$, because $D_x = 0$ outside $[0, 1]$. Probability mass in $[0, 1]$ is therefore conserved. In view of (15), gradient dynamics requires $v(0, t) = \phi_x(0, t)^+$ and $v(1, t) = \phi_x(1, t)^-$. For example, mass piles up at the endpoint 0 if (15) $v(0, t) = 0 > v(0+, t) := \lim_{x \downarrow 0} v(x, t)$ for some $t > 0$.

Technically and conceptually more difficult issues arise when the velocity field is discontinuous at an interior mass point. We will see in Section 8 that for certain payoff functions and initial conditions, probability conservation, as enforced by equations (12) and (13), results in a one-dimensional discontinuity set called a *shock wave*. To insure that $D(\cdot, t)$ remains a single-valued, cumulative distribution function, at such discontinuities we must look for a generalized or “weak” solution to the partial differential equation and impose the Rankine-Hugoniot conditions, e.g. ([Smo94], pp.245ff). To accommodate shock waves and to establish convergence results, we endow the set \mathcal{D} of distributions with the weak-* topology. This implies that the statement “ $D^{[n]}$ converges to D ” is equivalent to the statement “for each continuous function h on $[0, 1]$ we have $\lim_{n \rightarrow \infty} \int_0^1 h(y) dD^{[n]}(y) = \int_0^1 h(y) dD(y)$.”

5.3 Adjustment Costs

The basic LAP restriction that players’ actions do not jump can be motivated in many ways. Proposition 1 below shows that gradient dynamics emerge naturally when players respond optimally to current circumstances and face adjustment costs that increase quadratically in the adjustment speed.

We prepare for Proposition 1 with the following definitions.

◆A player located at $x(t)$ has *adjustment speed* $|v(x, t)| = \limsup_{h \rightarrow 0} h^{-1} |x(t+h) - x(t)|$. The velocity $v(x, t) = \dot{x}$ when the time derivative \dot{x} exists.

◆A player with payoff function $\phi(x, D)$ faces *quadratic adjustment cost* if for some constant $a > 0$ his net payoff at time t is $\phi[x(t), D(\cdot, t)] - a|v(x, t)|^2$.

◆Players are *myopically rational* if at each time $t \in (0, \infty)$ each player adjusts his current action $x(t)$ [at rate $\dot{x} = v(x, t)$] so as to maximize payoff net of adjustment cost over an arbitrarily short time horizon Δt .

Proposition 1 *Myopically rational players facing quadratic adjustment costs choose adjustment rates $v(x, t) = c\phi_x(x, t)$, where $c > 0$ is proportional to Δt for $x \in (0, 1)$ and satisfies the boundary conditions (15) at the endpoints $x = 0$ and 1.*

Myopic rationality in population games generally coincides with full rationality, in which players maximize the expected present value of the net payoff stream. This is because individual players cannot affect the population distribution and therefore cannot influence the future behavior of other players. Nevertheless, in special circumstances fully rational adjustment differs from myopically rational adjustment. These special circumstances include the availability of reliable information about payoffs at distant locations and about distant distributions, and a current distribution far from a stable invariant distribution.

5.4 Examples

Guessing game. In Keynes’ famous beauty contest the winning player guesses most closely the average opinion of all players. We consider a generalization in which the objective is to guess a multiple $a > 0$ of the average guess. The distributed payoff function is

$$\phi(x, D) = -0.5(x - a\bar{x})^2, \quad (16)$$

where $\bar{x} = \int x dD(x)$ is the mean action for D . Note that ϕ cannot be derived from a pairwise payoff function because it is nonlinear in D . In the Appendix we solve gradient dynamics explicitly for this game. As is shown there, with initial condition $F(x) = \rho(x, 0)$ the solution has density

$$\rho(x, t) = e^t F[xe^t + \bar{x}(0)(1 - e^{at})]. \quad (17)$$

As t increases the support of ρ becomes smaller and its density increases on its support. Thus we have clumping as $t \rightarrow \infty$. In particular, all mass converges exponentially to the initial mean $\bar{x}(0)$ if $a = 1$, to the lower endpoint 0 if $a < 1$, and to the upper endpoint 1 if $a > 1$. This behavior is consistent with the empirical results of [Nag95] for laboratory experiments with 15 – 18 paid human subjects, using a similar payoff function with $a = 2/3$.¹⁰

Bundt Payoff. Figure 3 shows the results of a numerical simulation of gradient dynamics using an initial uniform distribution and the “bundt” pairwise payoff function of Section 4.

Observe that clumping results, with all mass in the limit distribution concentrated at the points $z_{\pm} = .5 \pm .25/\sqrt{2}$ where the bundt rim $r = .25$ intersects the main diagonal $[x = y]$; the initial masses to the left and to the right of 0.5 are separately conserved for all $t \geq 0$. We will show in the next Section that this limit distribution is a Nash equilibrium, but that perturbations of the bundt payoff function lead to limit distributions that have only a local Nash property.

Replicator dynamics and inertial best-response dynamics produce different limiting distributions. Replicator dynamics depend sensitively on the initial distribution, because the set

¹⁰See also [Cra95] for an analysis of some related population game laboratory experiments with distributed payoff functions $\phi(x, D)$, in which the dependence on D is via an order statistic rather than via the mean action.

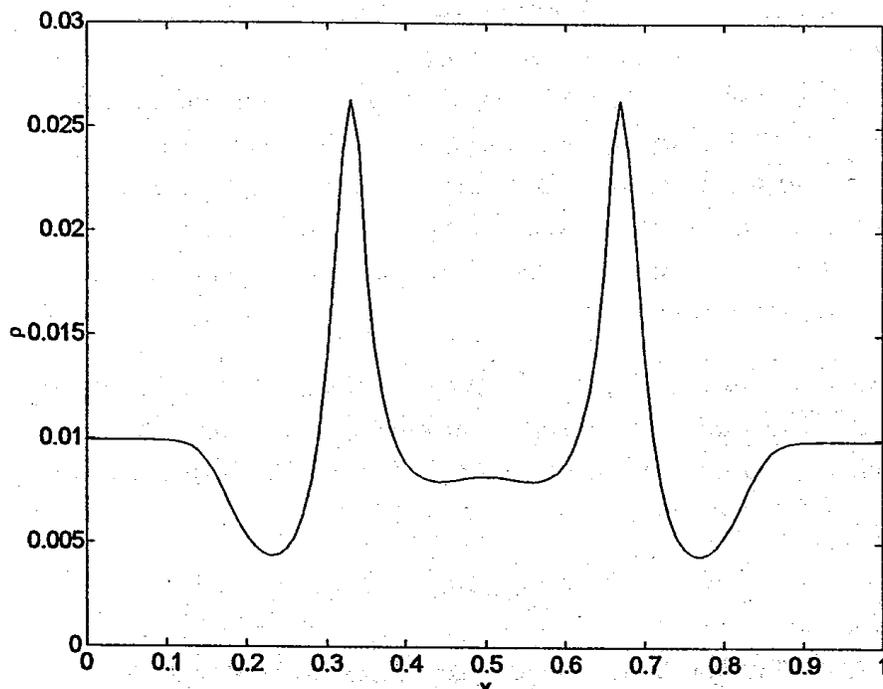


Figure 3: Simulation of gradient dynamics showing approximate asymptotic probability density associated with pairwise bundt payoff function $g(x, y) = 1024r^4(1 - r)\exp(-512r^4)$, where $r^2 = (x - .5)^2 + (y - .5)^2$.

of strategies actually employed does not expand as t increases. The limiting distribution for the bundt is concentrated on actions in the initial support as close as possible to the points z_{\pm} defined above. Inertial best response dynamics are much less dependent on the initial distribution. In the perturbed bundt example in Section 6.1 below, the limiting distribution has all mass at a single point, the Nash equilibrium, beginning from any initial distribution.

6 Invariant Distributions

In this section we explore more formally the geometric intuitions developed in Section 4 regarding steady state behavior. We begin with standard and new definitions pertaining to equilibrium.

◆ If D has a density ρ , the *support*, denoted $Supp(D)$, is the smallest closed set containing all points where ρ is positive.

The support of a distribution thus refers to the set of strategies in use. We also use the more general definition

◆ $Supp(D) = \{x \in [0, 1] : D(x + \epsilon) > D(x - \epsilon) \forall \epsilon > 0\}$.

◆ A *clumped* distribution has discrete support only. It is composed of mass points or *atoms*.

Such distributions are conveniently expressed in terms of the Heaviside step function $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$. For instance, the limiting distribution D^* in the bundt example (Cf. Fig.3) has $Supp(D^*) = \{z_-, z_+\}$, and we can write $D^*(x) = a\theta(x - z_-) + (1 - a)\theta(x - z_+)$, where $a = D(0.5, 0)$ is the total mass to the left of 0.5 in the initial distribution.

◆ *Dispersed* distributions have support containing an infinite set of points.

◆ A *Nash equilibrium* (NE) of a symmetric single population game ϕ is a distribution D in which every action in use is a best response.

In terms of the best response correspondence $B(D)$ defined in section 5.1, the condition for NE is $Supp(D) \subset B(D)$, or more explicitly, $\phi(y, D) \leq \phi(x, D)$ for all $y \in [0, 1]$ and all $x \in Supp(D)$.

◆ A *pure* NE is a NE distribution consisting of a single atom.

Hence the distribution $D(x) = \theta(x - z)$ is a pure NE if $\phi(x, D) \leq \phi(z, D)$ for all $x \in [0, 1]$. Much of the population game literature focusses on pure NE.

◆ A distribution D is a *local Nash equilibrium* (LNE) if $Supp(D)$ is discrete and the NE condition $\phi(y, D) \leq \phi(x, D)$ holds for all $x \in Supp(D)$ and all y sufficiently close to x .

In the bundt example, $\theta(x - z_-)$ and $\theta(x - z_+)$ are both pure NE, and any distribution of mass between the two points also is a NE. If we tilt the bundt slightly, say by adding $ax + by$ to $g(x, y)$ for small positive constants a and b , then the only NE is the pure NE supported at the upper point z_+ . The other equilibrium distributions become LNE. Such slight tilts have no qualitative effect on sign preserving dynamics. Hence LNE is a more robust equilibrium concept than pure NE when dynamics are sign preserving.

The equilibria of a dynamical process are steady states or *invariant distributions*.

◆ A distribution $D^* \in \mathcal{D}$ is invariant for LAP dynamics if the velocity field $v(x, t) = 0 \forall x \in SuppD^*$. A distribution $D^* \in \mathcal{D}$ is invariant under sign preserving (or gradient) dynamics for the distributed payoff function ϕ if

$$\phi_x(x, D^*) = 0 \forall x \in SuppD^*. \quad (18)$$

Given the boundary conditions (15), this condition can be relaxed to an inequality at the endpoints $x = 0, 1$. Furthermore, at points x of discontinuity of $\phi_x(\cdot, D^*)$ (18) becomes

$$\phi_x(x-, D^*) \geq 0 \geq \phi_x(x+, D^*). \quad (19)$$

◆ An invariant distribution D^* is called *stable* if it is locally asymptotically stable, i.e., if for every neighborhood V there is a neighborhood U such that every trajectory with initial distribution $D(\cdot, 0) \in U$ remains in V for all $t > 0$ and converges to D^* as $t \rightarrow \infty$.

6.1 Equal payoff property

Propositions 2 and 3 below show that NE and LNE distributions are invariant, and that invariant distributions locally provide an equal payoff for every action actually employed. Taken together, these results generalize the equal expected payoff property of mixed NE in standard games and, in the context of population games, generalize the Bishop-Cannings theorem ([MS82], p.182) to continuous action spaces.

Proposition 2 *Let D^* be a Nash Equilibrium or Local Nash equilibrium distribution for distributed payoff function ϕ . Then D^* is invariant under sign-preserving dynamics for ϕ .*

Proposition 3 *Let D^* be an invariant distribution under sign-preserving dynamics for distributed payoff function ϕ . Then $\phi(\cdot, D^*)$ is constant on every connected component of $\text{Supp}(D^*)$. If D^* is stable, then $\phi(\cdot, D^*)$ is maximized locally on every connected component of $\text{Supp}(D^*)$.*

Proposition 3 implies the landscape is flat over the support of a dispersed invariant distribution, and that the landscape is a “mesa” when the invariant distribution is stable.

7 Convergence to Equilibrium and the Idea of Progress

In this Section we show that an intuitive property of payoff functions we call *progressivity* ensures convergence under sign preserving dynamics to an invariant distribution. Thus there is no asymptotic divergence, convergence to a limit cycle, convergence to a strange attractor (chaos), or nonexistence of a distribution due to singular behavior over finite time.

◆ The *mean payoff* of a distribution $D \in \mathcal{D}$ for a given distributed payoff function ϕ is

$$\mu_D := \int \phi(x, D) dD(x) \tag{20}$$

In particular, if D has density ρ , and if ϕ has an underlying pairwise payoff function g , then we have the mean payoff

$$\mu_D(t) = \int_0^1 \int_0^1 g(x, y) \rho(x, t) \rho(y, t) dx dy. \quad (21)$$

◆ A distributed payoff function ϕ is *progressive* if its mean payoff $\mu_D(t) := \mu_{D(\cdot, t)}$ is strictly increasing under gradient dynamics except when $D(\cdot, t)$ is invariant.

Proposition 4 *Let ϕ be a progressive distributed payoff function. Assume sign preserving dynamics. Then, beginning from an arbitrary initial distribution, the trajectory converges asymptotically to some invariant distribution D^* . Moreover, if an isolated invariant distribution D^* is a strict local maximum of μ , then D^* is (locally asymptotically) stable.*

Remark: The main idea of the proof is that μ is a Ljapunov function for sign preserving dynamics when the payoff function is progressive. The result is significant because there are several types of games that are progressive and therefore always have convergent sign preserving dynamics.

◆ The pairwise payoff function g and the associated distributed payoff function $\phi = E[g]$ define a *game of common interest* if $g(x, y) = g(y, x)$. These are also known as *team games*.

Players in a game of common interest receive the same payoff for any given distribution, independent of their individual current actions. Thus all players have identical incentives to make the distribution more favorable.

◆ The distributed payoff function ϕ has the *nearest neighbor* (NN) property if it depends on the distribution only via its local value $D(x)$ or slope $D_x(x) = \rho(x)$.

◆ The NN payoff function ϕ defines a *basic congestion game* if $\phi(x, D)$ is a linear decreasing function of the density $D_x(x) = \rho(x)$ at x .

The payoff functions associated with hydrodynamics have the NN property.¹¹ The continuous analog of a standard congestion game, e.g. [Mil96], is an NN distributed payoff function which depends only on $\rho(x)$. Thus in a congestion game the payoff at any action x depends only on the fraction of the population that choose actions arbitrarily close to x .¹²

◆ An NN payoff function is *non-strategic* if the payoff depends on the current action x but not on the distribution D . In evolutionary biology, non-strategic payoffs result in *frequency independent selection*.

¹¹The NN restriction is important in models of random filling in statistical physics, physical chemistry and biochemistry. It is also crucial to numerical simulations of fluid flow in aerodynamics and the general study of turbulence, where it exemplifies the principle of no action at a distance. There are also some important examples of NN interaction in population biology (e.g., choice of territorial location) and in economics (e.g., choice of quality or location in Hotelling-type models).

¹²There is a closely related literature on potential games, e.g., [MS96], [Ros73]. The literature on congestion and potential games covers continuous as well as discrete action sets, but only for finite numbers of players. [MS96] consider inertial best response dynamics in finite action congestion games.

Proposition 5 *Let ϕ define one of the following: (a) a game of common interest; (b) a basic congestion game; (c) a nonstrategic game. Then ϕ is progressive, and the local maxima of μ are stable distributions under sign preserving dynamics for ϕ .*

7.1 Progress and evolution

Our definition of progressivity is consistent with the utilitarian notion that a society is better off the greater the sum total of individual utilities. Given an infinite population with mass scaled to 1, the mean payoff μ is then the natural measure of social well-being.

Beginning with [Dar59], many Victorian and some later writers argued that inherent in natural selection is ineluctable progress benefiting organisms and species.¹³ If only the fittest survive, the population becomes increasingly fit. If so, *all* payoff functions are progressive. This argument can be explored in the present context by studying the time derivative of (20), assuming for clarity that the density ρ exists:

$$\dot{\mu}(t) = \int_0^1 \phi(x, \rho) \rho_t(x, t) dx + \int_0^1 \frac{\partial}{\partial t} [\phi(x, \rho)] \rho(x, t) dx. \quad (22)$$

The first term on the right in (22) represents the direct effect of the population dynamics on the mean payoff, given an unchanging payoff environment. Using (14) and integrating by parts, we have

$$\int_0^1 \phi(x, \rho) \rho_t(x, t) dx = \int_0^1 \phi(\rho \phi_x)_x dx = \int_0^1 \phi_x^2 \rho dx \geq 0. \quad (23)$$

Equation (23) shows that the direct effect is the mean squared variation of the fitness function ϕ , which is strictly positive except at invariant distributions, where it is zero by theorem 3.

That Darwinian progress is not a general phenomenon is shown by the second term in (22), which represents the indirect effect on mean payoff of changes in the action distribution. The direct effect induces the players to climb towards higher ground. But the indirect effect moves the ground under their feet. There are special cases in which the direct effect dominates, consistent with Darwinian intuition. For example, in games of common interest and in basic congestion games, the indirect and direct effects both increase fitness. But in other games the indirect effect can be negative and dominate the direct effect.

Tragedy of the Commons. For example, let x be the exploitation intensity of a common resource such as pasture land, and choose pairwise payoff $g(x, y) = x - 2y$. The corresponding distributed payoff $\phi(x, D) = x - 2\bar{x}$, where \bar{x} is the spatial mean action for D . Then $\phi_x(x, D) = +1$, and each individual player gains by increasing x . On the other hand, the

¹³“And as natural selection works solely by and for the good of each being, all corporeal and mental environments will tend to progress toward perfection.” [Dar59]. See [Bur21], Chap. XIX, for a review of the post-Darwinian literature on progress and natural selection.

aggregate of such individual actions increases the mean \bar{x} , and, in view of $\phi_{\bar{x}} = -2$, harms every player more. We conclude that $\mu(t) = -\bar{x}$ is strictly *decreasing* except at a steady state because the indirect effect is negative and twice as strong as the direct effect. Ultimately, given any initial state, the distribution $D(\cdot, t)$ converges in finite time to the upper endpoint $D^* = \theta(x - 1)$. The result is that the pasture is overgrazed, and the mean payoff declines to its lowest possible value, $\mu = -1$.

8 Shock Waves and Status

We now consider NN payoff functions that implement rank or status dependence in the sense of [Veb99] or [Fra85]; the payoff at x depends only on the fraction of the population $D(x)$ using actions $y \leq x$. In such games, one's action (e.g., choice of house size or vehicle price) does not matter for its own sake and affects the player solely via its rank compared with all other players' actions. Rank payoff functions in general result in dynamics governed by nonlinear partial differential equations similar or identical to known equations in hydrodynamics. We show by example below that as in hydrodynamics, certain plausible initial conditions result in discontinuous shocks in densities or distributions.

◆ A *shock wave* in our context is a discontinuity in the density $\rho(x)$ or distribution $D(x)$ whose amplitude, $a(t) := D(x+, t) - D(x-, t) > 0$, and location $x = z(t) \in (0, 1)$, are continuous functions of time.

The status models below are characterized by shock waves of the compression type. An example of a compression shock is a sonic boom. In a population game, a compression shock in $D(x, t)$ marks the emergence and spread of clumping behavior at action $x = z(t)$, participated in by an initially growing fraction $a(t)$ of the population. One can interpret such an interior atom as a “middle class” whose action (e.g., consumption level or wealth) and size changes over time. Below, we give two relevant analytic solutions of the well known, first-order shock equation $u_t = c(u)u_x$ below.¹⁴

For simplicity, we take the *signal speed* $c(u)$ proportional to u . In our first example, u is the distribution D , and we have a solution of the shock equation composed of a uniform distribution with decreasing amplitude and a fixed shock represented by an atom at the endpoint $x = 1$. In the second example, u is the density ρ , and a right-moving compression shock develops at finite time.

Cusp payoff. We consider a population game in which every player earns the difference between his current position and all lesser positions. Then the pairwise payoff $g(x, y) =$

¹⁴From a study of the characteristics of this equation, one observes [Lax72] that if the *signal speed* $c(u)$ is monotone increasing in u , and if the spatial derivative of the initial wave function $u_x(x, 0) := \partial u_0(x)/\partial x < 0$ on some open interval, then a spatial discontinuity develops in $u(x, t)$. This discontinuity in u appears at finite time and thereafter propagates with a speed set by conditions insuring mass conservation.

$(x - y)\theta(x - y)$. The basic gradient dynamics equation (14) reads

$$\rho_t(x, t) = -[\rho(x, t) \int_0^1 \theta(x - y)\rho(y, t)dy]_x. \quad (24)$$

In (24), the partial derivative $g_x(x, y) = \theta(x - y)$, in view of the fact that the generalized function $x\delta(x) = 0$. In terms of the cumulative distribution $D(x, t) = \int_0^x \rho(y, t)dy$, (24) therefore becomes

$$D_t = -D_x D, 0 \leq x < 1, \quad (25)$$

with boundary condition $D(1, t) = 1$. Equation (25) is a simple form of the mass conservation equation for the one-dimensional flow of an inviscid fluid. As is well known, a continuous solution of (25) can be written in the parameterized form

$$D = D_0(x - Dt), \quad (26)$$

where the initial condition is $D(x, 0) = D_0(x)$. For illustrative purposes, we set $D_0(x) = x$, so that we have the initial uniform density $\rho_0(x) = 1$ on $[0, 1]$. Inserting this initial condition into (26), the full solution of (25) becomes

$$D(x, t) = \frac{x}{1+t} + \frac{t}{1+t}\theta(x-1). \quad (27)$$

Step payoff. Consider a second population game in which the players earn a fixed amount proportional to the number of players at lesser positions. That is, $\phi(x, D) = D(x)$ is precisely the rank or status of the player in the distribution. Then the pairwise payoff is $g(x, y) = \theta(x - y)$, and (25) is replaced by the almost identical relation

$$\rho_t = -2\rho\rho_x \quad (28)$$

for the density ρ . We write the parameterized solution

$$\xi = x - 2\rho t, \quad \rho = \rho_0(\xi), \quad (29)$$

with initial condition $\rho(x, 0) = \rho_0(x)$. As before, the continuous solution is

$$\rho = \rho_0[x - 2\rho t], 0 \leq t \leq t^* \quad (30)$$

which holds on a range of t to be determined.

We choose as initial condition the inverted parabola

$$\rho_0(x) = 48x\left(\frac{1}{2} - x\right), 0 \leq x \leq \frac{1}{2}, \quad (31)$$

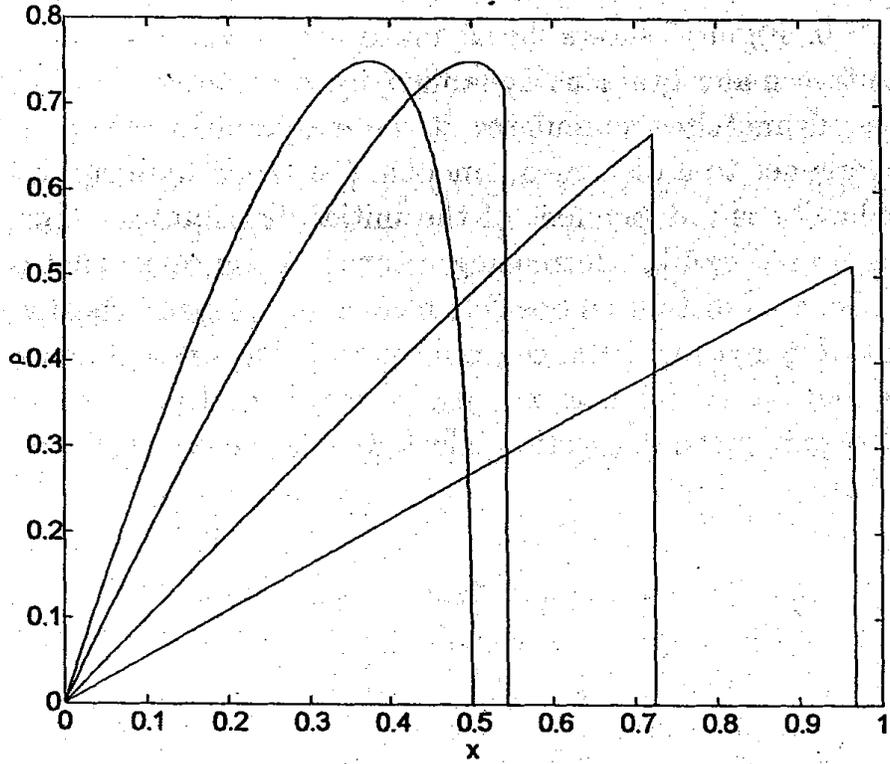


Figure 4: Time development of shock wave in probability density of actions, given cusp pairwise payoff function $g(x, y) = (x - y)\theta(x - y)$.

with support only on $[0, 1/2]$. Because (31) is monotone decreasing on $[1/4, 1/2]$, a shock front develops at $x = 1/2$ and moves to the right until all probability mass piles up at $x = 1$. As shown in the Appendix, the strength and position of the front are given, respectively, by

$$v(\xi_-) = \frac{3(\sqrt{1 + 3\tau} - 2)}{(\sqrt{1 + 3\tau} - 1)^2}, \quad (32)$$

$$z_s(\tau) = \frac{\frac{7}{3} + 2\tau - \frac{5}{3}\sqrt{1 + 3\tau}}{\sqrt{1 + 3\tau} - 1}. \quad (33)$$

The movement of the shock wave is shown in Figure 4.

9 Dispersal

Many of the examples presented so far — compression shock waves, migration towards an endpoint, and concentration at an interior local NE — feature clumping behavior. The next

two examples demonstrate the opposite sort of behavior, in which the action distribution becomes more dispersed with time, in the extreme case becoming the uniform distribution in $[0, 1]$.

Expansion wave. We note first the classic case of the centered expansion wave or “fan” (see, e.g., [Log94], Chapt. 3). We assume again the pairwise cusp payoff function $g(x, y) = (x - y)\theta(x - y)$, leading to $D_t = -D_x D$, $0 \leq x < 1$, as in equation (25). However, we choose the initial condition $D_0(x, 0) = \theta(x - q)$, so that initially there is an atom at some interior point $q, 0 < q < 1$. The known solution of $D_t = -D_x D$ under these conditions is

$$D(x, t) = \theta(x - q)\left[\theta(t - x + q)\frac{x - q}{t} + \theta(x - q - t)\right], \quad t < 1 - q. \quad (34)$$

Equation (34) tells us, in terms of the probability density D_x , that for all times t such that $0 < t < 1 - q$, the initial atom disperses into a square wave of width t and amplitude $1/t$, extending over the x -interval $[q, q + t]$. For all times $t > 1 - q$, probability mass piles up at $x = 1$.

Symmetric power-law payoff. Consider the pairwise payoff function $g(x, y) = |x - y|^a$ where $a > 0$. Figure 5 shows the asymptotic behavior beginning from an initial uniform distribution under gradient dynamics, for $a = .5$ and $a = 2$, respectively. In general, all mass ultimately accumulates at the endpoints $x = 0$ and 1 when $a > 1$, but we get convergence to a dispersed, smooth, U -shaped limiting distribution when $a < 1$. This behavior is independent of the initial distribution. The reason is that payoff functions g or ϕ exhibit decreasing returns in the sense that a player has an incentive to choose x as distant as possible from other players’ choices. For $a > 1$ the marginal benefit of increasing distance from other active strategies is greater for more distant actions, but for $a < 1$ nearby actions of other players are more important. Thus, dispersal occurs given congestion effects in the sense of sufficiently strong *local* decreasing returns.

9.1 Analysis

We now show that if payoffs are sufficiently impaired by congestion there is asymptotic convergence to a dispersed invariant distribution. We formalize the idea of congestion effects (i.e., strong locally decreasing returns) and prove that the resulting invariant distribution has no atoms.

◆ For given $K > 1$, the distribution D is K -denser at x than at y if there is some $\hat{\epsilon} > 0$ such that $D(x + \epsilon) - D(x - \epsilon) > K[D(y + \epsilon) - D(y - \epsilon)] \forall \epsilon \in (0, \hat{\epsilon})$.

Examples: (i) the (unit) normal distribution is K -denser at 0 than at 1 for all $K < \exp(-0.5) \approx 1.6$. (ii) If D has an atom at x but not at y , then it is K -denser at x than at y for all positive K .

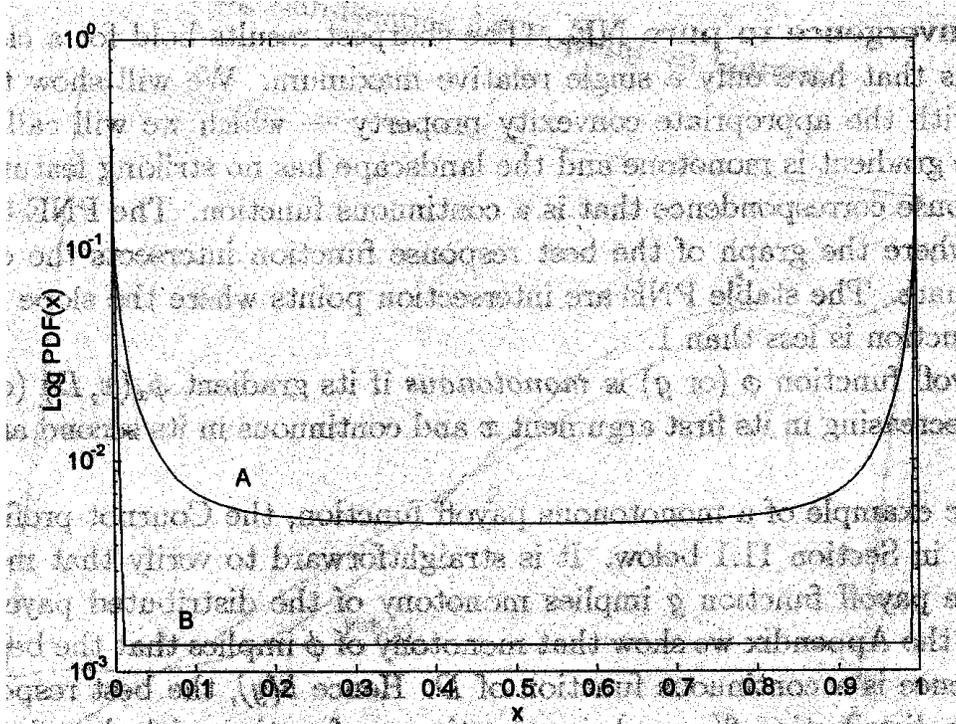


Figure 5: Approximate asymptotic probability densities for pairwise payoff function $g(x, y) = |x - y|^a$. Curve A: $a = .5$. Curve B: $a = 2$.

◆ The distributed payoff function ϕ is *locally repulsive* if there is some $K > 1$ such that $\phi(x, D) < \phi(y, D)$ whenever D is K -denser at x than at y .

Examples: (i) if $a < 1$ then $\phi(x, D) = \int |x - y|^a dD(y)$ is locally repulsive. (ii) Basic congestion games (Cf. sec. 7) in which the payoff decreases with $\rho(x)$ are locally repulsive.

Proposition 6 *Let D^* be an invariant distribution under sign-preserving dynamics for a locally repulsive distributed payoff function ϕ . Then D^* has no atoms.*

10 Clumping

In this section we find conditions on payoff functions that ensure complete clumping under sign preserving dynamics. We consider convergence to pure Nash equilibria (PNE) and also consider some related equilibrium concepts in evolutionary biology.

10.1 Convergence to pure NE

The sharpest results hold for a class of payoff functions that have only a single relative maximum. We will show that payoff functions with the appropriate convexity property — which we will call *monotony* because the gradient is monotone and the landscape has no striking features — have a best response correspondence that is a continuous function. The PNE turn out to be points where the graph of the best response function intersects the diagonal of the unit square. The stable PNE are intersection points where the slope of the best response function is less than 1.

◆ A payoff function ϕ (or g) is *monotonous* if its gradient $\phi_x(x, D)$ (or $g_x(x, y)$) is strictly decreasing in its first argument x and continuous in its second argument D (or y).

A classic example of a monotonous payoff function, the Cournot profit function, is discussed in Section 11.1 below. It is straightforward to verify that monotony of the pairwise payoff function g implies monotony of the distributed payoff function $\phi = Eg$. In the Appendix we show that monotony of ϕ implies that the best response correspondence is a continuous function of D . Hence $b(y)$, the best response to the pure atomic distribution $\theta(x - y)$, is a continuous function with domain and range $[0, 1]$.

The next result characterizes pure Nash equilibria for our systems as intersections of the graph of b with the diagonal $[x = y]$ in the unit square, and shows that monotony is a sufficient condition for existence of pure Nash equilibrium. It is natural to identify a pure NE, $D^*(x) = \theta(x - y^*)$, with its support y^* , and to write $y^* \in PNE(\phi)$.

Proposition 7 $PNE(\phi) = Diag \cap Gr[b]$, and if ϕ is monotonous then $PNE(\phi) \neq \emptyset$.

We now consider the stability of PNE.

◆ The PNE $y^* \in Diag \cap Gr[b]$ is a *downcrossing* if, for y sufficiently close to y^* , the difference $(b(y) - y)$ has the same sign as $(y^* - y)$. The PNE is an *upcrossing* if $(b(y) - y)$ has the opposite sign from $(y^* - y)$.

If b is differentiable at y^* , then we have a downcrossing (upcrossing) if $b'(y^*) < 1$ (> 1). See Figure 6 for an illustration.

It is well known that for “generic” functions $b(\cdot)$ every PNE is either an upcrossing or a downcrossing (see, e.g., [GP74]). Then by the intermediate value theorem there are $2n - 1$ PNE, where $n \geq 1$ is the number of downcrossings.

The main result of this subsection is that PNE are the only invariant distributions for monotonous payoff functions, and downcrossings support the only stable distributions.

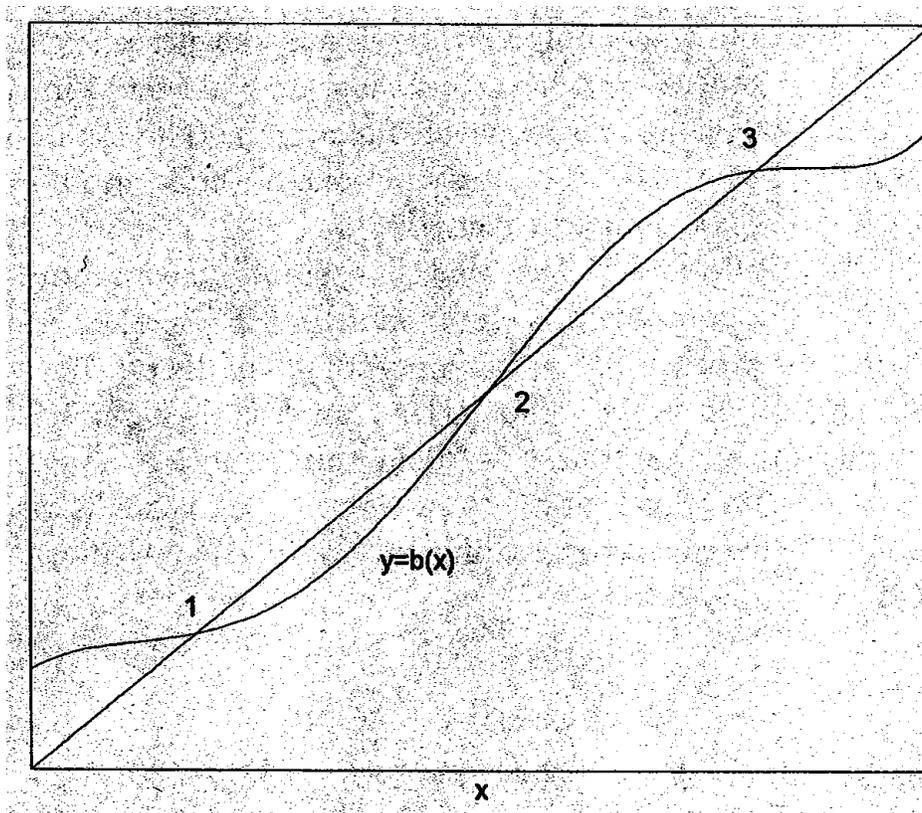


Figure 6: Best response $y = b(x)$ given action x , showing stable Nash Equilibrium downcrossings (1,3) and unstable upcrossing (2).

Proposition 8 *Let D^* be an invariant distribution under sign preserving dynamics for a monotonous distributed payoff function ϕ . Then D^* is a pure Nash equilibrium. If D^* is also stable, then it is supported at a downcrossing. At least one downcrossing exists.*

10.2 ESS and CSS

Following [MSP73], there has been considerable discussion of points in action space called evolutionarily stable strategies (ESS). Clumping at these points is a stable configuration in the sense that an ESS resists invasion by any other strategy carried out by a small fraction of the population. The normal context is a two-player symmetric game with a finite unordered action set. [EM81] and several later authors point out that the conditions for an ESS require strengthening when the action set is continuous. [Esh83] argues for an equilibrium concept called *continuously stable strategy* or CSS, which resists small deviations by the entire population. In this subsection, we show that the key CSS inequality condition is essentially the same as the downcrossing concept introduced above in Section 10.1, and that applying the relevant downcrossing condition implies stability against more general

perturbations than are considered in the literature.

◆Let $\phi = Eg$ for a given pairwise payoff function g . The action $x^* \in [0, 1]$ is an *ESS* for g if $\phi(y, D) < \phi(x^*, D)$ for all distributions of the form $D(x) = (1 - \epsilon)\theta(x - x^*) + \epsilon\theta(x - y)$ for all $\epsilon > 0$ sufficiently small.

The strict inequality above is intended as a stability condition. It is based on the intuition that if the population is clumped at an ESS x^* then a small invasion of mutants clumped at any other action (phenotype) y will die out because the mutants receive a lower payoff. It is well known that an ESS is a pure NE. This follows immediately if one takes the limit $\epsilon \rightarrow 0$ in the definition above, recovering the defining property of a pure NE that x^* is a best response to the distribution $\theta(x - x^*)$.

If the payoff g is twice continuously differentiable one can write the local necessary conditions for ESS as

$$g_x|_{x=x^*}(x, x^*) = 0; \quad (35)$$

$$g_{xx}|_{x=x^*}(x, x^*) \leq 0. \quad (36)$$

These local conditions are also sufficient if the fitness function is monotonous and the inequality in (36) is strict.

Conditions (35,36) do not guarantee stability against small perturbations of the continuous phenotype [EM81]. If the entire clumped distribution is displaced slightly from an ESS x^* , an additional condition is needed to ensure local stability, in the sense that the distribution moves back towards x^* . The gradient at x must be positive (negative) when x is slightly below (above) x^* . In view of (35), we can write the additional condition as

$$(d/dx|_{x=x^*})g_x(x, x) = g_{xx}(x^*, x^*) + g_{xy}(x^*, x^*) \leq 0. \quad (37)$$

[Esh83] shows that if (37) holds with strict inequality at an ESS x^* , then x^* satisfies the following definition of continuous stability.

◆The phenotype $x \in [0, 1]$ is a *CSS* for the symmetric two-player game g if $\exists \epsilon > 0 \forall y \in (x - \epsilon, x + \epsilon) \exists \delta > 0 \forall z \in (y - \delta, y + \delta) [g(z, y) > g(y, y) \iff |z - x| < |y - x|]$.

This formal definition implicitly assumes that the phenotype distribution remains clumped (at y or z), contrary to the motivation of CSS as a stability concept.

Proposition 9 *Let g be a twice continuously differentiable pairwise fitness function, and let D^* be a clumped distribution supported at a point $x^* \in [0, 1]$. If D^* is a NE stable under sign preserving dynamics for $\phi = Eg$, then conditions (35 - 37) hold. Conversely, if conditions (35 - 37) hold with strict inequalities, then D^* is a pure NE, and x^* is a downcrossing.*

Hence, stability implies the local conditions (35 - 37), and these conditions with strict inequalities are sufficient for stability. Note that our results permit arbitrary transient distributions,

more general monotonous fitness functions ϕ (not necessarily additive random-pairwise from a smooth g), and arbitrary sign preserving dynamics.

11 Applications

11.1 Cournot Duopoly

The classic Cournot duopoly model, e.g., [FT91], posits two firms that simultaneously choose outputs x and y . This model is static. It is assumed that these firms have zero fixed cost and identical constant marginal cost and face a downward sloping linear demand function. For convenience, scale the marginal cost to 0 and also scale the downward slope and intercept of demand to 1. Then price is $1 - x - y$, and the restriction $x, y \in [0, 1]$ is natural. The pairwise payoff function to the firm choosing x when its rival chooses y is then $g(x, y) = x(1 - x - y) = \text{profit} = \text{revenue}$.

We now place the Cournot model in a dynamic context. Either of two interpretations of the distributed payoff

$$\phi(x, D) = \int_0^1 x(1 - x - y)dD(y). \quad (38)$$

is a suitable starting point.

The first interpretation is that each firm faces a large number of potential rivals whose output choices have distribution D . Firms have strategic uncertainty in the sense that they have no knowledge of D . Nevertheless, as they marginally adjust output they see on average the local profit gradient $\phi_x(x, D)$.

The second interpretation, based on [Sky86], is that the Cournot payoff represents the subjective expected profit of a firm contemplating (but not yet committed to) output x . The notion is that management's anticipations of their rival's future choices of actions are summarized in D . Gradient or other dynamical adjustment models then represent an internal process of modifying beliefs as the firm contemplates the potential profit consequences to itself and its rivals. Only when the process converges to an invariant distribution does the firm actually commit to produce output.

Under either interpretation (38) holds, and the payoff function has gradient

$$\phi_x := v(x, t) = \int_0^1 (1 - 2x - y)\rho(y, t)dy = 1 - 2x - \bar{x}(t). \quad (39)$$

We assume the initial distribution has density $F(x) := \rho(x, 0)$, with mean $\bar{x}(0)$. Thus the continuity equation (14) becomes

$$\rho_t = -v_x\rho - v\rho_x = 2\rho - [1 - 2x - \bar{x}(t)]\rho_x. \quad (40)$$

The initial value problem for (40) can be solved using the same approach as in the guessing game of Section 5.4. From (40), we find that the mean action satisfies the differential equation $d\bar{x}/dt = 1 - 3\bar{x}$. If the solution $\bar{x}(t) = [\bar{x}(0) - 1/3] \exp(-3t) + 1/3$ is inserted into (40), we obtain the linear partial differential equation

$$\rho_t = 2\rho - \left[\frac{2}{3} - 2x + (\bar{x}(0) - \frac{1}{3})e^{-3t}\right]\rho_x. \quad (41)$$

Standard techniques yield the solution

$$\rho(x, t) = e^{2t} F\left[\bar{x}(0) + \left(x - \frac{1}{3}\right)e^{2t} - \left(\bar{x}(0) - \frac{1}{3}\right)e^{-t}\right]. \quad (42)$$

Inspection of (42) shows that as t increases, the initial support shrinks at exponential rate e^{2t} toward the limit distribution $\delta(x - \frac{1}{3})$, which is a pure Nash equilibrium. That $\rho(x, t)$ converges smoothly to an asymptotic equilibrium is in fact a general property that follows from the results in Section 10. In particular, it is easy to check that $g(x, y)$ is monotonous. The results of Section 10 further show that the assumption of an initial dispersed distribution $\rho(x, 0)$ is superfluous. We conclude that in its extension to population games the standard, single atom Nash-Cournot equilibrium is even more robust than might have been supposed. Even with a continuum of potential firms and with sign preserving dynamics the pure Nash equilibrium is globally stable.

11.2 The Edgeworth-Bertrand Model as Foraging Competition

We now consider a population game based on the Edgeworth-Bertrand (EB) duopoly model [Edg97]. The EB model has been well studied as an example of capacity constrained price competition and as a model of resource use in population biology (e.g. [Rog95]). We address the question whether cycling or mixed NE behavior occurs in the context of gradient dynamics for the EB population game. Different views appear in the literature. Recent discussions of capacity constrained price competition, (e.g., [Tir88], pp.211ff. and p.234), argue for a mixed NE rather than cycles. However, these models include no dynamics. In the population biology literature the suggestion has been made [Rog95] that there are cycles in certain circumstances.¹⁵ [Edg97] also argued for a cyclic outcome given price competition in which duopolistic firms are capacity constrained, so that neither firm alone can meet demand at zero price. Near price 0, it is profitable for one firm to withdraw from the market and then re-enter at a high price to meet residual demand. Edgeworth argued that if this occurs the other firm will also raise its price, and the two firms will again compete on price, gradually driving the price down until one firm withdraws and later reenters at high price, so that prices cycle. As we shall explain, our numerical simulations suggest the conclusion that with gradient dynamics there is asymptotic, monotonic convergence to a mixed NE.

¹⁵In particular [Rog95] argues that replicator dynamics cycle if the cost parameter c introduced below is less than 0.5, and that this value is a cutoff above which replicator dynamics leads to an asymptotic ESS .

We begin with equivalent economic and biological interpretations of the EB model.

Duopoly. Two firms compete to supply a market. At any time, each firm may choose to participate actively or not to participate. Non-participation earns zero profit. The two firms play a pairwise game, using price as their action. Let $x \in [0, 1]$ denote the price charged by an active firm, while $x = Z$ denotes non-participation. An active firm incurs a positive fixed cost c and has zero marginal cost. All consumers buy a single unit from the lowest cost firm up to a choke price of 1. When both firms charge the same price, consumers split their total demand evenly. We scale maximal demand to 1.

Under these conditions the pairwise payoff function is

$$\begin{aligned}
 g(Z, y) &= 0, \\
 g(x, y) &= x - c, \quad x \in [0, 1], \quad y > x, y = Z \\
 g(x, y) &= 0.5x - c, \quad y = x \in [0, 1] \\
 g(x, y) &= -c, \quad 0 \leq y < x \leq 1.
 \end{aligned} \tag{43}$$

Harvesting. There is an equivalent biological interpretation. A resource has gross value 1 when fully ripe and a continuum of values x at earlier times, scaled so that $x \in [0, 1]$. By incurring a cost $c \in [0, 1)$, a player has the right to try to harvest at his chosen time $\tau = x$, but another player has the same opportunities, and the player choosing the earliest harvest time gets the crop. Non-participation, denoted $x = Z$, is another possible action. The payoff function is then precisely the EB $g(x, y)$ in (43).¹⁶

We now apply gradient dynamics to the EB population game. The definition of a distribution needs to be generalized slightly to accommodate the non-standard action space $A = [0, 1] \cup \{Z\}$. As before, let $D(x, t)$ represent the fraction of the population at time t choosing actions in the interval $[0, x]$, where $x \in [0, 1]$. We denote the non-participating fraction of the population by $\rho(Z) = 1 - D(1) \geq 0$. The expected payoff then becomes

$$\begin{aligned}
 \phi(Z, D) &= 0, \\
 \phi(x, D) &= -c + x(1 - D(x)), \quad x \in [0, 1].
 \end{aligned} \tag{44}$$

Gradient dynamics itself also requires some generalization in the context of this non-standard action space. No change in (12) is required at points (x, t) where $\phi(x, D(\cdot, t)) > 0$ and

¹⁶The biological interpretation in the text is based on the [Rog95] model of foraging competition. There are further interesting interpretations. [Rog95] implicitly relates his model to the tragedy of the commons and explicitly to sealed bid common value auctions with an entry fee, citing [LS94]. Reinterpreting the action variable as maximal waiting time instead of price, the model becomes a version of the war of attrition; see [FT91], especially chapter 4.5. Models of competition between species where size has a cost (but is advantageous in direct competition) raise many of the same issues, e.g., whether cycles are relevant [AM94].

$D(1, t) = 1$. At other points, probability mass can flow from points of negative distributed payoff in $[0, 1]$ to Z , and from Z to points in $[0, 1]$ with positive payoff.

EB invariant distribution. An invariant distribution D^* satisfies the equation $\phi(x, D^*) = 0$ for all x in $Supp D^*$. We state without proof that the invariant distribution is unique.¹⁷ $D^*(x) = 0$ for actions with $x < c$ that are associated with negative profit and are therefore dominated by non-participation $x = Z$. Inserting (44) into $\phi(x, D^*) = 0$ for $c \leq x \leq 1$, we have the continuous action space equilibrium density $D^*(x) = 1 - c/x$, with mass $1 - D^*(1) = c = D^*(Z)$. The associated continuous density $\rho^*(x) = c/x^2$ for $x \geq c$ and $\rho^*(x) = 0$ for $c \leq x \leq 1$.

Discrete model. For computational purposes, we specify the EB dynamics on a discrete action space. Let $j = 0, \dots, n - 1$ index grid points between 0 and 1. Let $j = n$ index the no-participation point $x = Z$. At $j = n$ the pairwise payoff function $g(n, i) = 0$, so the (vectorial) distributed payoff $\phi(Z, D) = 0$. On the other hand, at regular grid points $k \geq 0$, the distributed payoff is $\phi(k, D) = -c + (1 - D_k)k/(n + 1)$, where $D_k = \sum_{j=0}^{k-1} \rho_j + 0.5\rho_k$ is the probability that a player with current action k is preempted by another player's lower choice j . As before, the dynamics are generated by outflow from points k to all nearest neighbors that have higher ϕ , at rates proportional to the differences $\phi_{k\pm 1} - \phi_k, \phi_n - \phi_k$.

The novelty here is an extra nearest neighbor at the no-participation point $j = n$. This extra lattice point can have zero mass and always has zero payoff. The evolution matrix (3) therefore takes the form

$$\mathbf{M} = \begin{pmatrix} X_1^- + \phi_0^- & X_1^+ & \dots & 0 & 0 & \phi_0^+ \\ -X_1^- & -X_1^+ + X_2^- + \phi_1^- & \dots & 0 & 0 & \phi_1^+ \\ 0 & -X_2^- & \dots & 0 & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \\ 0 & 0 & \dots & X_{n-2}^+ & 0 & \dots \\ 0 & \dots & \dots & -X_{n-2}^+ + X_{n-1}^- + \phi_{n-2}^- & X_{n-1}^+ & \phi_{n-2}^+ \\ 0 & 0 & \dots & -X_{n-1}^- & -X_{n-1}^+ + \phi_{n-1}^- & \phi_{n-1}^+ \\ -\phi_0^- & -\phi_1^- & \dots & -\phi_{n-2}^- & -\phi_{n-1}^- & -\sum_j \phi_j^+ \end{pmatrix} \quad (45)$$

with an extra row and column associated with the non-participation option.

EB Simulation. We have simulated the EB game for different choices of initial density $\rho(0)$ and cost c . As shown in Figure 7 for the particular choice of an initial uniform density and $c = .3$, our numerical analysis suggests the system asymptotically progresses monotonically to equilibrium, whatever the initial state.¹⁸

¹⁷Similar arguments to the ones used in Proposition 3 can be used to support this conclusion for the EB non-standard action space.

¹⁸A numerical consistency check on monotonicity is given by following the time derivative of the scalar ρ^2

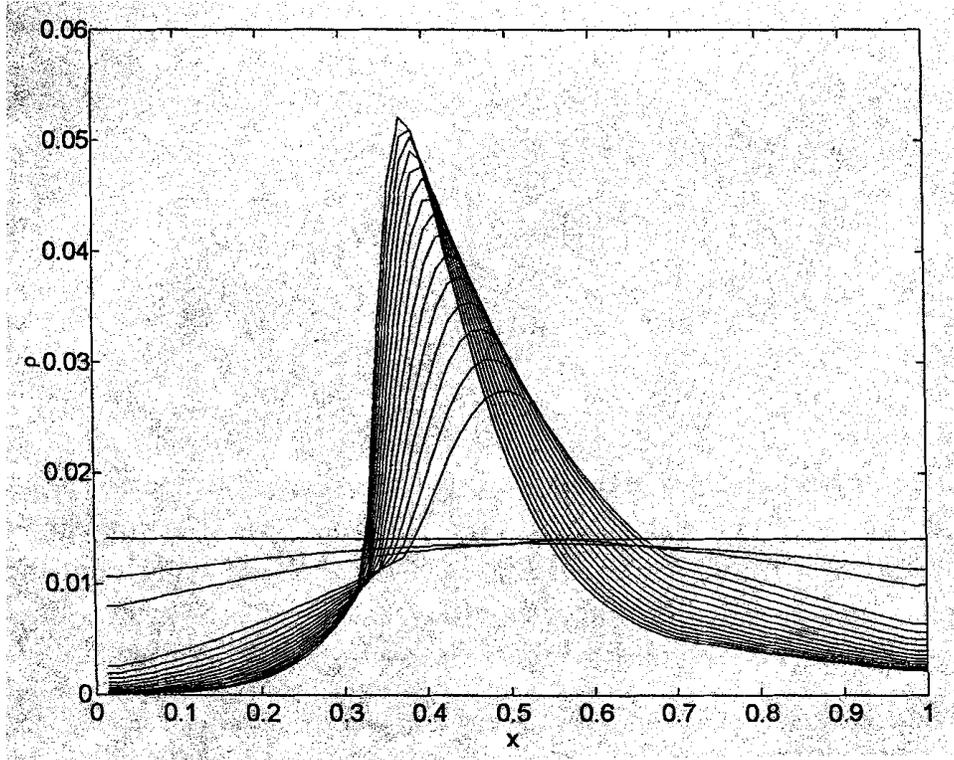


Figure 7: Simulation of gradient dynamics for the Edgeworth-Bertrand (or Foraging) model, with parameter $c = 0.3$ (see text). From a uniform distribution $\rho(0)$, there is monotonic progression towards the asymptotic density $\rho^*(x)$ given in the text, which has a step discontinuity at $x = c$.

11.3 Relation of LAP to other models

We conclude this section with some brief remarks on other economic and biological models in the literature from the perspective of LAP dynamics.

11.3.1 Location of Firms

Firm location is perhaps the most obvious economic application of our population games. A salient example is the “main street” model of [Hot29], in which firms choose location x on

along a trajectory. We have

$$\frac{d\rho^2}{dt} = \rho^T(\mathbf{M} + \mathbf{M}^T)\rho := V,$$

where T indicates transpose. The quantity $V = 0$ at equilibrium, because negative profits force components $\rho_k = 0$, $k \leq Nc$, and the equilibrium payoff condition $\phi_k = 0$, $k > Nc$, which implies $\mathbf{M} = \mathbf{0}$ for evolution matrix elements describing positive profit transitions. We find V goes monotonically to 0 near equilibrium for every type of initial condition.

the line segment $[0, 1]$, delivered price increases with distance from buyer to supplier, and demand goes to the firm offering the lowest delivered price. Clumping of suppliers results in the most popular versions of the model.

Following [ASS83] (who extend the work of [Son82]), consider the Hotelling-type model in which profit is earned by a firm at location x on the infinite line \mathcal{R} when the distribution of firm locations is D . Supply at x is the density $\rho(x, t)$ of the current distribution $D(\cdot, t)$. Demand arises from buyers with a fixed location distribution who each buy a single unit at the lowest full price. Full price to a buyer at distance r from x is the price at x plus r^2 . The distributed payoff function ϕ is defined implicitly by the prices determined by the condition that supply equals demand at each infinitesimal location x . The supply distribution $D(x, t)$ adjusts according to gradient dynamics, essentially as in our equation (14).¹⁹

The main conclusion of [ASS83] is that the supply distribution converges to a dispersed invariant distribution when demand is dispersed. The basic feature that leads to this result is that prices (and hence profit) are lower at locations with a lower ratio of demand to supply. It is straightforward to show that this feature implies that the payoff function is locally repulsive on the support of the demand distribution. Consequently our Proposition 6 immediately implies the conclusion of [ASS83]. This derivation suggests that convergence to a dispersed distribution, as in [ASS83], is more natural than the conventional conclusion that firms will clump their actions.

11.3.2 Evolution of Continuous Biological Traits

Models of gradient selection have been widely used to describe the evolution of quantitative characters (e.g. gestation time, foot speed, beak size, etc.) For example, given the normal distribution of phenotype associated in many instances with a polygenic trait, one can argue that the mean phenotype evolves at a rate proportional to the gradient of the fitness function,

$$\frac{d\bar{x}}{dt} = \sigma^2 \phi_{\bar{x}}(\bar{x}), \quad (46)$$

where σ^2 is the additive genetic variance [Lan82]. More generally, the dynamics reflect *frequency dependent* selection where fitness $\phi(x, D)$ depends on the distribution D of a trait across the current population. Most existing models assume fitness depends on D only via summary statistics. For example, in recent discussions of evolutionary stability and ESS

¹⁹Our results show that one need not assume gradient dynamics; sign preserving dynamics will do. Nor is it necessary to assume transportation costs that are quadratic in distance. The [ASS83] analysis implicitly relies on the fact that interactions are assumed nearest neighbor (NN). The competitive price at a point x responds to changes in the the distribution D at points y far away from x , but it turns out that only the total mass to either side of x matters, so indeed interactions are NN with respect to the cumulative distribution function. Using our methods, one can show that this property is inessential also. We note also that we are able to work on the original state space $[0, 1]$, while [Son82] and [ASS83] avoid endpoint problems by working on the unit circle and infinite line respectively.

[e.g., [AMH93], [Tay95]] the summary statistic used is the spatial mean $\bar{x} = \int x dD(x)$, with a dynamics of the form

$$\bar{x}_t = \sigma^2 \phi_x(x, \bar{x}). \quad (47)$$

.Our approach offers three extensions of the existing models. First, we are able to consider more general forms of frequency dependence, consistent with general theory [Wri49]. For example, this generality permits examining a range of models of maladaptation [Lan76] and coevolutionary stability [AMH93]. Models assuming D is concentrated on a single point \bar{x} , or on two or more discrete points [MGM⁺95], can be extended to a continuous trait distribution. A particular case occurs in Darwin finches where a subset of the population with very similar beak size has important evolutionary significance [Wei94]. It would be natural to model fitness $\phi(x, \cdot)$ in that case to depend on the distribution D via its density $\rho(x)$ at a particular point rather than on the spatial mean \bar{x} . Second, we have results for sign-preserving dynamics, not just strict gradient dynamics. Sign preserving dynamics can deal with non-constant σ^2 . Third, while the existing literature deals with static concepts such as ESS and CSS, our results hold for a variety of static and dynamic equilibria, e.g. locally asymptotically stable invariant distributions.

Consider the following example adapted from [Esh83], example 1, which is a slight generalization of the quadratic model that obtains in weak stabilizing selection [LS96]. Let $g(x, y; z) = -(x - z)^2 - (x - y)^2$, $0 < z < 1$, so that the fitness of phenotype x increases quadratically in the distance from an exogenously optimal value z and from the phenotype y of other individuals. Then one can verify from (21) that the payoff function is progressive. Hence, by Proposition 4 there is a unique limit distribution, and that distribution maximizes μ . We conclude that $D^*(x) = \theta(x - z)$ is the global attractor from any initial distribution.

This example exhibits an extension of Eshel's conclusion that inefficient ESS's for the discrete version of the model disappear when the trait is continuous. Thus, it appears one need not impose Eshel's strong assumption that the initial trait distribution is completely clumped. Our approach can also deal with much more general specifications of the fitness value of gregariousness, such as quadratic distance from the population mean (as in Sect. 5.4.2), or the fraction of the population within a fixed distance interval ϵ .

12 Summary and Discussion

We have argued here that local adjustment path (LAP) dynamics, which permit adjustment with only finite velocity, are natural for population games with an ordered action space. LAP dynamics are distinct from replicator, best reply and other dynamics previously studied in the evolutionary games literature. LAP dynamics unify separate literatures on nearest neighbor (NN) interactions from fluid dynamics, mean field interactions from population biology and elsewhere, and random pairwise interactions from evolutionary game theory. LAP dynamics can be visualized as taking place on a payoff landscape that shapes the continuous adaptation of the player population and is itself reshaped by the adapting population.

Our main analytic results concern the population distribution of player behavior (and the corresponding landscapes) on the unit interval $[0, 1]$ for sign-preserving dynamics, in which the local adjustment path has the same direction as the payoff gradient. We have shown how such dynamics arise in the presence of adjustment costs, and have characterized the landscapes of their invariant distributions. We are able to associate LAP invariant distributions with well-known static equilibrium concepts such as Nash equilibrium (NE) and evolutionary stable strategy (ESS). Sufficient conditions were developed for convergence to invariant distributions, and for convergence to clumped distributions and to dispersed distributions. We have described applications in economics and biology and have exhibited the results of numerical simulations.

Many important theoretical issues remain open. To our knowledge, little formal work has as yet been done to explore continuous action spaces for non-LAP dynamics. We conjecture that non-LAP models, such as replicator dynamics, have properties on continuous action spaces similar to their known behavior on discrete, finite action spaces. We expect that more sophisticated analysis is possible for the relation between adjustment costs and the implied dynamics. Implicit adjustment costs arise from risk or aversion to ambiguity - e.g. in circumstances under which each player knows the payoff and the payoff gradient at his current location, but is uncertain of the current distribution D (strategic uncertainty) and may not even know the payoff function beyond the immediate neighborhood of his current action. We conjecture further that if uncertainty, as measured by the variance of the estimate of $\phi(x + z, D)$, increases linearly in $|z|$ then gradient dynamics will result. For more general forms of adjustment cost, we expect to see sign-preserving dynamics.

It remains an open question whether sign preserving dynamics necessarily converge to an invariant distribution for non-progressive payoff functions. We have found no counterexamples thus far. The proof of Proposition 4 would work just as well if one replaced the mean fitness μ by any other functional that increases along trajectories. We conjecture that some appropriate measure of entropy or the negative of mean squared variation can serve as a Lyapunov function for a broad class of payoff functions. We believe it also is possible to find sharper characterizations of payoff functions that always converge to clumped distributions or dispersed distributions.

The arguments presented here allow for considerable generalization. One can deal with population games among m distinct (sub)populations or types $k = 1, \dots, m$ that have different action sets A_k . The relevant set of distributions is the joint probability measures $\mathcal{D}(A_1 \times \dots \times A_m)$. It is also straightforward to construct asymmetric population games from an arbitrary m -player game in normal form by assuming random matching across other subpopulations and taking expectations.

Richer action spaces than $[0, 1]$ can also be accommodated. In fact, intervals unbounded on either or both sides are more tractable than a bounded interval. We have already expanded the action space somewhat in the Edgeworth/foraging and firm location applications of Section 11. Higher dimensional action spaces present no conceptual obstacles, although there we expect dynamic equilibria to include cycles and chaotic attractors, as well as the

point attractors (stable distributions) we have studied here. The discrete approximations used here also extend in a natural fashion. The set of nearest neighbors is larger — $2D$ in D dimensions — but the uphill flow rules still apply to each pair of nearest neighbors and can be implemented directly.

We believe that new applications of LAP dynamics offer research opportunities. There are many physiological and behavioral traits subject to frequency dependent selection. LAP dynamics are appropriate when a single mutation typically changes the trait values only slightly. In economics, politics, and finance one often has interactions among many actors with essentially continuous choices, e.g., the location of firms, the capital structure or degree of financial leverage, the incentive intensity of employment contracts, the degree of revelation in bidding at auction, or credit standards. Given explicit or implicit adjustment costs, LAP dynamics is a natural way to model these applications.

13 Appendix

13.1 Notation

Given a function f of two variables x and y , and some fixed value of y , $f(\cdot, y)$ denotes the function of the single variable x whose value is $f(x, y)$.

$A := B$ means “ A is defined as B .”

$Gr[f]$ refers to the graph $\{(x, y) \in [0, 1]^2 : y = f(x)\}$ of a function f that has domain and range $[0, 1]$.

$Diag := \{(x, y) \in [0, 1]^2 : x = y\}$ denotes the diagonal in the unit square.

Trailing $+$ and $-$ refer respectively to left and right limits, *viz.* $h(0+) = \lim_{x \downarrow 0} h(x)$.

Superscripted $+$ and $-$ refer respectively to positive and negative truncations, *viz.* $x^+ = \max\{0, x\}$, $x^- = \min\{0, x\}$.

The Heaviside step-function, $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$, can be used to write $x^\pm = x\theta(\mp x)$.

The Dirac delta-function $\delta(x)$ is the (improper) density associated with the cumulative distribution function $\theta(x)$, *viz.* $\theta'(x) = \delta(x)$.

We denote partial derivatives by subscripts, e.g. $f_x(x, t) = \frac{\partial f}{\partial x}$.

We drop arguments of functions, e.g. x for $x(t)$, when the meaning is clear in context.

Diamonds (◆) label definitions; squares (■) indicate the completion of a proof; triangles (▲) indicate the completion of a derivation.

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13.2 Proofs

PROPOSITION 1. Myopically rational players facing quadratic adjustment costs choose adjustment rates $v(x, t) = c\phi_x(x, t)$, where $c > 0$ is proportional to Δt for $x \in (0, 1)$ and satisfies the boundary conditions (15) at the endpoints $x = 0$ and 1 .

PROOF OF PROPOSITION 1: Consider a player who adopts current strategy $x(t) \in (0, 1)$ and faces quadratic adjustment cost at rate av^2 , for some $a > 0$. The objective function then is

$$\text{Max}_v \int_0^{\Delta t} (\phi(x + vs, D(\cdot, t + s)) - av^2) ds.$$

For Δt small we have the trapezoidal approximation for the maximand

$$0.5(\phi(x, D(\cdot, t)) + \phi(x + v\Delta t, D(\cdot, t + \Delta t)))\Delta t - av^2\Delta t.$$

Since by assumption a single player has negligible influence on the population distribution $D(\cdot, t + s)$, the first-order condition is $0 = 0.5\phi_x\Delta t - 2av$ with solution $v = 0.25\Delta ta^{-1}\phi_x$. The second-order condition automatically is satisfied for $a > 0$ and Δt sufficiently small. Hence at interior points the conclusion holds with $c = .25\Delta ta^{-1} > 0$. At endpoints we have the same maximization problem with the additional constraint $v \geq 0$ at $x = 0$ or $v \leq 0$ at $x = 1$. The conclusion follows. ■

PROPOSITION 2. Let D^* be a Nash Equilibrium or Local Nash equilibrium distribution for distributed payoff function ϕ . Then D^* is invariant under sign-preserving dynamics for ϕ .

PROOF OF PROPOSITION 2: Let D^* be NE or LNE for payoff function ϕ , and let $x \in \text{Supp}D^*$. By definition, $\phi(\cdot, D^*)$ is locally maximized at x . Thus, if it exists, the gradient $\phi_x(x, D^*) = 0$. If $\phi_x(x, D^*)$ does not exist, then the relevant inequality conditions (19) are satisfied. Hence D^* is invariant. ■

PROPOSITION 3. Let D^* be an invariant distribution under sign-preserving dynamics for distributed payoff function ϕ . Then $\phi(\cdot, D^*)$ is constant on every connected component of $\text{Supp}(D^*)$. If D^* is also stable, then $\phi(\cdot, D^*)$ is maximized locally on every connected component of $\text{Supp}(D^*)$.

PROOF OF PROPOSITION 3: Let $x \in \text{Supp}(D^*)$ be isolated. Any function defined only on a singleton is trivially constant, so it suffices for an isolated point to show that if x is not a local maximum of ϕ then D^* is unstable. Suppose now that x is not a local maximum. Then in any neighborhood N of x there is a y such that $\phi(y, D^*) > \phi(x, D^*)$. For specificity, say $y > x$. If the gradient ϕ_x is continuous on $[x, y]$, then by the Mean Value Theorem there is some $z \in (x, y)$ such that $\phi_x(z) > 0$. By continuity in D , this inequality holds for all D of the form $(1 - a)D^* + a\tilde{D}$, where $0 \leq a \leq \epsilon$, and \tilde{D} is a distribution with support containing $[x, y]$. For any sign preserving dynamics beginning from an initial distribution of such form with $D(z, 0) < D^*(z)$, we see directly from sign preservation that $D(z, t)$ is decreasing in t . Hence the trajectory $D(\cdot, t)$ fails to converge to D^* as $t \rightarrow \infty$, so D^* indeed is unstable. The case $y < x$ is handled in a similar fashion: one finds a point $z \in (y, x)$ with $\phi_x(z, D^*) < 0$ and shows that mass to the left of z increases over time, so that the distribution under sign preserving dynamics again fails to converge to the atomic distribution D^* . The continuity assumption on ϕ_x is not essential. By taking a sufficiently close continuous approximation of the gradient the same argument follows. Arbitrarily close continuous approximations are available because the continuous functions are a dense subset of the piecewise continuous functions on $[0, 1]$ by Lusin's Theorem (e.g., [Rud73], p 53).

Next consider an interior point x of an interval, I , contained in $\text{Supp}(D^*)$. The invariance of D^* implies that $v(x, D^*) = 0$ and sign preservation implies that $\phi_x(x, D^*) = 0$ for every x in the interval. Hence the payoff is constant on the interval I , i.e., on any connected component of S . The local maximization argument for a connected component is essentially the same as for an isolated point. That is, one takes x to be the lower or upper endpoint of a support component. If x is not a relative maximum, one finds a nearby point z and an initial distribution that smears a small amount of probability mass past z , and shows that the smeared mass never returns to the support component. ■

LEMMA 1. The partial differential equation (12) with boundary conditions (15) and initial condition has a unique solution $D(\cdot, t)$ for all $t > 0$. The solution depends continuously on the initial condition $D_0 \in \mathcal{D}$.

PROOF OF LEMMA 1. We offer no independent proof of Lemma 1, but instead refer the reader to [Smo94] for an existence and uniqueness result. It is straightforward but tedious to extend that proof, which uses finite difference methods, to the present case. Our boundary conditions (15) simplify some steps in the proof. ■

PROPOSITION 4. Let ϕ be a progressive distributed payoff function. Assume sign preserving dynamics. Then, beginning from an arbitrary initial distribution, the trajectory converges asymptotically to some invariant distribution D^* . Moreover, if an isolated invariant distribution D^* is a strict local maximum of μ , then D^* is (locally asymptotically) stable.

PROOF OF PROPOSITION 4: Let $\mathcal{T} = \{D(\cdot, t) : t \geq 0\}$ be the trajectory under given sign preserving dynamics beginning from an arbitrary point $D(\cdot, 0) \in \mathcal{D}$. Lemma 1 guarantees that

\mathcal{T} is a well defined subset of \mathcal{D} . By the Banach-Alaoglu Theorem (e.g., [Rud73], p.66), \mathcal{D} is compact in the weak-* topology. Hence \mathcal{T} has a limit point $D^{**} \in \mathcal{D}$. Suppose, contrary to the Proposition, that D^{**} is not invariant. Since ϕ is progressive and the dynamics are sign preserving, μ is strictly increasing on the trajectory beginning at D^{**} . Hence for some time $t_0 > 0$ and some $\epsilon > 0$, the mean payoff exceeds $\mu(D^{**}) + \epsilon$ for all $t > t_0$. By the continuity of μ in D (following from the definitions of the weak*-topology and of μ) and the continuity of points on a trajectory in the initial condition (from Lemma 1), we see that μ achieves the value $\mu(D^{**}) + \epsilon/2$ by time t_0 on a trajectory beginning at a point of \mathcal{T} sufficiently close to D^{**} , say at $D(\cdot, t_1)$. Now progressivity implies that μ is bounded below by $\mu(D^{**}) + \epsilon/2$ on the tail of \mathcal{T} . In particular, at the limit point D^{**} we have $\mu(D^{**}) \geq \mu(D^{**}) + \epsilon/2$, a contradiction. Hence D^{**} is invariant. Now suppose that D^* and D^{**} are both limit points of \mathcal{T} . The previous argument shows that both points are invariant and $\mu(D^*) = \mu(D^{**})$. The continuity of the solution in initial conditions (from Lemma 1) now implies that the distance between D^* and D^{**} can be made arbitrarily small. Therefore $D^* = D^{**}$, i.e., the trajectory \mathcal{T} converges asymptotically to D^* .

For the last part of the Proposition, let D^* be a strict local maximum and take its neighborhood V small enough that it contains no other invariant distributions or local maxima. Since μ is continuous, it attains a maximum $k < \mu(D^*)$ on the boundary of V . Let $U = V \cap \{D \in \mathcal{D} : \mu(D) > k\}$. Progressivity already implies that no trajectories beginning in U ever leave V . To complete the proof, we need only establish asymptotic stability, i.e., that D^* is the unique limit point of any trajectory beginning in U . But the argument of the previous paragraph establishes that any limit point D^{**} must be invariant. By construction, D^* is the only invariant distribution in V . Hence $D^{**} = D^*$. ■

PROPOSITION 5. Let ϕ define one of the following: (a) a game of common interest; (b) a basic congestion game; (c) a nonstrategic game. Then ϕ is progressive, and the local maxima of μ are stable distributions under sign preserving dynamics for ϕ .

PROOF OF PROPOSITION 5: (a) Let ρ be a density function with $\rho(0, t) = \rho(1, t) = 0$. We need only show that the function

$$\mu(t) = \int_0^1 \int_0^1 g(x, y) \rho(x, t) \rho(y, t) dx dy$$

is strictly increasing except at invariant distributions, because distributions defined by such density functions are a dense subset of \mathcal{D} . Using the evolution equation (14) and integrating by parts, the time derivative of μ is

$$\dot{\mu} = \int_0^1 \int_0^1 \int_0^1 g_x(x, y) [g_x(x, z) + g_x(z, x)] \rho(x, t) \rho(y, t) \rho(z, t) dx dy dz, \quad (48)$$

where the surface terms from the parts integrals are zero.

For symmetric $g(x, y)$ as in a game of a common interest, the terms in the integrand of (48) are equal, and we may write

$$\dot{\mu} = 2 \int_0^1 \rho(x, t) \left[\int_0^1 g_x(x, y) \rho(y, t) dy \right]^2 dx \geq 0, \quad (49)$$

with strict inequality except when $E_D g = 0$ on $Supp \rho = Supp D$. By Proposition 3 we have strict inequality except at an invariant distribution, and the payoff function therefore is progressive.

(b) The proof for a basic congestion game is similar. Note that basic congestion games can be scaled so that $\phi(x, D(\cdot, t)) = -\rho(x, t)$. Thus we have $\phi_x = -\rho_x$, and $(\partial/\partial t)\phi(x, D(\cdot, t)) = -\rho_t = [\phi_x \rho]_x = -[\rho_x \rho]_x$, using the continuity equation (14) for the next to last equality. Hence the second term in (22), the "indirect effect," is

$$\int_0^1 (\partial/\partial t)\phi(x, D(\cdot, t)) \rho(x, t) dx = - \int_0^1 [\rho_x \rho]_x \rho(x, t) dx = \int_0^1 [\rho_x \rho] \rho_x dx = \int_0^1 \rho \rho_x^2 dx \geq 0, \quad (50)$$

where we integrate by parts in the central equality. Recalling that $[\rho_x(x, t)]^2 = \phi_x^2$, we conclude that the indirect effect as well as the direct effect is positive except at invariant distributions, where it is zero. Hence the payoff function is progressive.

(c) In a non-strategic game, ϕ is independent of $D(\cdot, t)$, and we have $(\partial/\partial t)\phi(x, D(\cdot, t)) = 0$. Hence the second term in (22) is zero and $\dot{\mu}$ is given by (23). As already noted, (23) is positive except at invariant distributions. Hence the payoff function is progressive. ■

PROPOSITION 6. Let D^* be an invariant distribution under sign-preserving dynamics for a locally repulsive distributed payoff function ϕ . Then D^* has no atoms.

PROOF OF PROPOSITION 6: Suppose to the contrary that the limit D^* has an atom of mass $a > 0$ at $x \in [0, 1]$. Since the total mass is finite, there are points y arbitrarily close to x at which D^* has no atoms, and D^* is a fortiori K -denser at x than at such y for any $K > 1$. By the locally repulsive property, $\phi(\cdot, D^*)$ is greater at such y than at x . Hence ϕ does not achieve a local maximum on its support, so by Proposition 3 we conclude that D^* is not invariant. ■

LEMMA 2. Suppose ϕ is monotonous. Then for all $D \in \mathcal{D}$:

- (a) The function $\phi(\cdot, D)$ has a unique maximum $x^* = B(D) \in [0, 1]$,
- (b) B is continuous on \mathcal{D} , and
- (c) $sgn \phi_x(y, D) = sgn[B(D) - y]$ for all $y \in [0, 1]$.

PROOF OF LEMMA 2: Redefine $B(D)$ as $\sup\{x \in [0, 1] : \phi_x(x, D) > 0\}$. Then B inherits its continuity in D from ϕ_x . Since $\phi_x(\cdot, D)$ is strictly decreasing, it must be > 0 (resp. < 0) for $y < B(D)$ (resp. $y > B(D)$). Thus (b) and (c) are immediate, and it suffices for (a) to show that for $y \neq x^* = B(D)$ we have $\phi(x^*, D) > \phi(y, D)$. Suppose to the contrary that $0 \leq \phi(y, D) - \phi(x^*, D)$. If ϕ_x is continuous then by the mean value theorem, the right hand

side is $(y - x^*)\phi_x(z, D)$ for some z between y and x^* . But the last expression is the product of two non-zero terms of opposite sign when $y \neq x^*$, by part (c). Hence the expression is negative, a contradiction that proves the result. Discontinuities in ϕ_x can be dealt with by smooth approximations, as in the proof of Proposition 3. ■

PROPOSITION 7. $PNE(\phi) = Diag \cap Gr[b]$, and if ϕ is monotonous then $PNE(\phi) \neq \emptyset$.

PROOF OF PROPOSITION 7: The geometric characterization $PNE = Diag \cap Gr[b]$ is immediate from the definitions. To prove existence, note that $b(y)$, and hence $b(y) - y$, are continuous functions by the previous lemma. We have $b(y) - y \geq 0$ at $y = 0$ and $b(y) - y \leq 0$ at $y = 1$. Hence, by the intermediate value theorem $b(y) - y$ has at least one root y^* . Thus, every monotonous distributed payoff function has at least one downcrossing $y^* \in PNE$. ■

LEMMA 3. Let $y^* \in PNE$ be a downcrossing, and let $D(x, 0) = \theta(x - y)$. For y sufficiently close to y^* , the distribution $D(x, t)$ converges to $\theta(x - y^*)$ as $t \rightarrow \infty$ under sign preserving dynamics for a monotonous payoff function.

PROOF OF LEMMA 3: In the case $y < y^*$ we have $y < b(y)$ since y^* is a downcrossing and thus $\phi_x(y, \theta(\cdot - y)) > 0$ by Lemma 2. For sign preserving dynamics we then have $v(y, t) > 0$. Indeed, the definition of sign preservation implies that for $t > 0$ the support of $D(\cdot, t)$ is contained in the interval $[y(t), y^*]$, where the lower bound $y(t) > y(0) = y$ increases in t at a rate commensurate with $\phi_x(y^*, \theta(\cdot - y^*)) > 0$. Hence the distribution converges to $\theta(\cdot - y^*)$ as $t \rightarrow \infty$. The case $y > y^*$ is entirely analogous. ■

PROPOSITION 8. Let D^* be an invariant distribution under sign preserving dynamics for a monotonous distributed payoff function ϕ . Then D^* is a pure Nash equilibrium. If D^* is also stable, then it is supported at a downcrossing. At least one downcrossing exists.

PROOF OF PROPOSITION 8: Let D^* be invariant. By Lemma 2(c) we have $\phi_x < 0$ for $y > B(D^*)$ and $\phi_x > 0$ for $y < B(D^*)$. Hence by Proposition 3, the support of D^* consists of the single point $B(D^*)$. From the definition of B in the proof of Lemma 2 we see that D^* is indeed a PNE supported at $x^* = B(D^*)$. If D^* is stable, then by Lemma 3 x^* must be a downcrossing. The last part of the proof of Proposition 7 established the existence of at least one downcrossing. ■

PROPOSITION 9. Let g be a twice continuously differentiable symmetric two-player fitness function, and let D^* be a clumped distribution supported at a point $x^* \in [0, 1]$. If D^* is a NE stable under sign preserving dynamics for $\phi = Eg$, then conditions (35 - 37) hold. Conversely, if conditions (35 - 37) hold with strict inequalities, then D^* is a pure NE, and x^* is a downcrossing.

PROOF OF PROPOSITION 9: Suppose D^* is an NE stable under sign preserving dynamics for $\phi = Eg$. The Nash property implies that $g(\cdot, x^*)$ is maximized at $x = x^*$, so by smoothness

we have (35 - 36). By Proposition 8 we know that x^* is a downcrossing. Hence, $1 > b'(x^*) = -g_{xx}(x^*, x^*)/g_{xy}(x^*, x^*)$, where the last equality follows from the implicit function theorem and the characterization of b as the locus $g_x(x, x) = 0$. But cross multiplication of the last inequality yields (37). Conversely, suppose conditions (35 - 37) hold with strict inequalities. The first two conditions imply that the Nash property holds locally, hence (by monotony) $x^* \in PNE(\phi)$. Given (37), it follows immediately that $1 > b'(x^*)$, and we conclude that x^* is a downcrossing. ■

13.3 Derivations

1. Derivation of Equation (6).

We write (2,3) as an autonomous nonlinear system in the vector of variables ρ . Define the discrete derivative

$$\mathbf{S} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \quad (51)$$

By the definitions of the expected payoff $\varphi = \mathbf{g}\rho$ and the first differences $\mathbf{X} = \mathbf{S}\varphi$, we have $\mathbf{X} = \mathbf{S}\mathbf{g}\rho$. Equation (2) therefore takes the autonomous form

$$\dot{\rho} = \mathbf{M}(\mathbf{S}\mathbf{g}\rho) \cdot \rho. \quad (52)$$

By construction, the pairwise payoff $\mathbf{g} = \mathbf{S}^{-1}$, so that $\mathbf{X} = \mathbf{S}\mathbf{g}\rho = \rho$. Given $\mathbf{X} = \rho$, only the terms \mathbf{X}^+ contribute in (3), and the evolution matrix in (52) is given by

$$\mathbf{M} = \begin{pmatrix} 0 & \rho_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\rho_1 & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & \dots & \rho_k & 0 & 0 & \dots \\ 0 & 0 & \dots & -\rho_k & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \rho_{n-1} & 0 \\ 0 & \dots & 0 & \dots & \dots & -\rho_{n-1} & \rho_n \\ 0 & 0 & 0 & 0 & 0 & 0 & -\rho_n \end{pmatrix}. \quad (53)$$

Substituting (53) into (52) gives the coupled Riccati system (6).

2. *Solution of (6) for $n=2$.* This system is solvable in the three-component case $n = 2$. From (6), we have, by inspection,

$$\rho_2(t) = \frac{1}{t + \frac{1}{\rho_2(0)}}. \quad (54)$$

The Riccatti equation $\dot{\rho}_1 = -\rho_1^2 + \rho_2^2$ integrates to

$$\rho_1(t) = \rho_2(t) \frac{\alpha_+ + k\alpha_- [\rho_2(t)]^{\sqrt{5}}}{1 + k[\rho_2(t)]^{\sqrt{5}}}, \quad (55)$$

where $\alpha_{\pm} = (1 \pm \sqrt{5})/2$, and

$$k = -\rho_2(0)^{-\sqrt{5}} \frac{\rho_1(0) - \alpha_+ \rho_2(0)}{\rho_1(0) - \alpha_- \rho_2(0)}. \quad (56)$$

By probability conservation, $\rho_0(t) = 1 - \rho_1(t) - \rho_2(t)$.▲

3. Derivation of equation (17).

Equation (16) yields the gradient $\phi_x = a\bar{x}(t) - x$. The continuity equation then becomes

$$\rho_t = -[(a\bar{x}(t) - x)\rho]_x. \quad (57)$$

To solve (57), first linearize by taking the spatial expectation of both sides, obtaining, after straightforward algebra,

$$d\bar{x}/dt = (a - 1)\bar{x}, \quad (58)$$

which gives

$$\bar{x}(t) = \bar{x}(0)e^{(a-1)t}. \quad (59)$$

Inserting (59) into (57), we have the linear partial differential equation

$$\rho_t = \rho + [x - a\bar{x}(0)e^{(a-1)t}] \rho_x, \quad (60)$$

which can be solved by standard textbook techniques, see, e.g., [Cop75]. In particular, the general solution of (60) takes the form

$$\rho = e^t F[k(x, t)], \quad (61)$$

where $F(k)$ is a function of k fixed by the initial condition $F(k) = \rho(k, 0)$, with spatial mean $\bar{x}(0)$. On inserting (61) into (60), we observe that the function $k(x, t)$ may be chosen to be any particular solution of

$$k_t = [x - a\bar{x}(0)e^{(a-1)t}]k_x, \quad (62)$$

with $k(x, 0) = x$. We choose $k(x, t)$ to be a solution of the characteristic equation $dx/dt = x - a\bar{x}(0)e^{(a-1)t}$, viz.

$$k(x, t) = xe^t + \bar{x}(0)(1 - e^{at}). \quad (63)$$

Substituting (63) into (61), we confirm that the solution to (57) is equation (17) of the text.▲

4. Derivation of equations (32, 33).

For algebraic convenience we define the new variables $z = 4x - 1$, $v = 4\rho$, $\tau = 3t$. In terms of these quantities, the new normalized density $v(z, \tau)$ is defined on the interval $[-1, 3]$, with initial support on $[-1, 1]$. Equations (28-31) now read

$$v_\tau = -\frac{2}{3}vv_z \quad (64)$$

$$\xi = z - \frac{2}{3}v\tau, \quad v = v_0(\xi), \quad (65)$$

$$v = v_0\left(z - \frac{2}{3}v\tau\right), \quad 0 \leq \tau \leq \tau^* \quad (66)$$

$$v_0(z) = \frac{3}{4}(1 - z^2), \quad -1 \leq z \leq 1; \quad v_0(z) = 0, \quad 1 \leq z \leq 3. \quad (67)$$

From (66) and (67) the continuous solution for the density is

$$v(z, \tau) = \frac{-1 + \tau z + \sqrt{1 - 2\tau z + \tau^2}}{\frac{2}{3}\tau^2}, \quad 0 \leq \tau \leq 1 \quad (68)$$

A shock wave develops at the earliest time for which the expression under the radical in (68) becomes zero. At that time the spatial derivative v_z becomes singular. The shock occurs at $z = 1$, the right-hand edge of the initial support, at time $\tau = \tau^* = 1$.

For $\tau > 1$ the inviscid wave “breaks,” and the density becomes multi-valued with no clear meaning. One can retain, nevertheless, a single valued density for $\tau > 1$ by allowing a weak solution with a shock discontinuity moving to the right. The standard technique [Lax72] is to impose mass conservation on the discontinuity via the equal area, Rankine-Hugoniot condition

$$\frac{1}{2} [v_0(\xi_+) + v_0(\xi_-)] (\xi_+ - \xi_-) = \int_{\xi_-}^{\xi_+} v_0(\xi) d\xi, \quad (69)$$

as well as a pair of continuity conditions on the medium, which here read

$$z_s = \xi_+ = \xi_- + \frac{2}{3}v_0(\xi_-)\tau. \quad (70)$$

Since there is partial initial support, $v_0(\xi_+) = 0$, and on using $v_0(\xi_-) = \frac{3}{4}(1 - \xi_-^2)$, (69) becomes

$$\frac{1}{2}v_0(\xi_-)(\xi_+ - \xi_-) = \int_{\xi_-}^1 v_0(\xi)d\xi. \quad (71)$$

Carrying out the integral in (71) using the initial condition (67),

$$\xi_+ - \xi_- = \frac{\frac{2}{3}(1 - \xi_-)(2 + \xi_-)}{1 + \xi_-}. \quad (72)$$

From (70) and (72), the shock occurs at time

$$\tau(\xi_-) = \frac{4}{3} \frac{(2 + \xi_-)}{(1 + \xi_-)^2}. \quad (73)$$

Inverting (73), we have

$$\xi_-(\tau) = \frac{2}{3\tau} \left(1 + \sqrt{1 + 3\tau}\right) - 1. \quad (74)$$

It follows that the shock has strength

$$v(\xi_-) = \frac{3(\sqrt{1 + 3\tau} - 2)}{(\sqrt{1 + 3\tau} - 1)^2}, \quad (75)$$

and position

$$z_s(\tau) = \frac{\frac{7}{3} + 2\tau - \frac{5}{3}\sqrt{1 + 3\tau}}{\sqrt{1 + 3\tau} - 1}, \quad (76)$$

as given in the text.

Alternative derivation of (73). One may also impose probability conservation directly, viz.

$$\frac{\partial}{\partial \tau} \int_{-1}^{z_s(\tau)} v(z, \tau) dz = 0. \quad (77)$$

Because the position of the shock is $z_s = \xi_+$, and its strength is $v(z_s, \tau) = v_0(\xi_-)$, we have, using $v_\tau = -\frac{1}{3}(v^2)_z$ (Cf.(64)) in (77),

$$\frac{dz_s}{d\tau} = \frac{1}{3}v_0(\xi_-). \quad (78)$$

On the other hand, differentiating (70) gives

$$\frac{dz_s}{d\tau} = \frac{d\xi_-}{d\tau} + \frac{2}{3}v_0(\xi_-) + \frac{2}{3}v'_0(\xi_-)\tau\frac{d\xi_-}{d\tau}. \quad (79)$$

From (78) and (79),

$$\frac{d\xi_-}{d\tau} = -\frac{1}{4}\frac{(1 - \xi_-^2)}{1 - \tau\xi_-}, \quad (80)$$

which is a form of Abel's differential equation of the second kind. On exchanging dependent and independent variables, we obtain a linear inhomogeneous ODE for $\tau(\xi_-)$. This integrates easily, and we again obtain (73). \blacktriangle

References

- [AM94] Peter Abrams and Hiroyuki Matsuda. Evolution of traits that determine ability in competitive contests. *Evolutionary Ecology*, 8:667–681, 1994.
- [AMH93] Peter Abrams, Hiroyuka Matsuda, and Yasushi Harada. Evolutionarily unstable fitness maxima and stable fitness minima of continuous traits. *Evolutionary Ecology*, 7:465–487, 1993.
- [ASS83] Phillipe Artzner, Carl Simon, and Hugo Sonnenschein. *Convergence of Myopic Firms to Long-Run Equilibrium via the Method of Characteristics*, volume 264 of *Lecture Notes in Economics and Mathematical Systems*, pages 157–183. Springer-Verlag, Berlin Heidelberg New York Tokyo, 1983.
- [Bin87] Kenneth Binmore. Modeling rational players part 1. *Economics and Philosophy*, 3:179–214, 1987.
- [BS94] Kenneth Binmore and Lawrence Samuelson. Muddling through noisy equilibrium selection. *unpublished manuscript*, 1994.
- [Bur21] John B. Bury. *The idea of progress; an enquiry into its origin and growth*. Macmillan and Co., London, 1921.
- [Cop75] E. T. Copson. *Partial Differential Equations*. Cambridge University Press, London, 1975.

- [Cra95] Vincent Crawford. Adaptive dynamics in coordination games. *Econometrica*, 63:103–144, 1995.
- [Dar59] Charles Darwin. *On the origin of species by means of natural selection*. J. Murray, London, 1859.
- [EA83] I. Eshel and E. Akin. Coevolutionary instability of mixed nash solutions. *J. Math. Biol.*, 18:123–134, 1983.
- [Edg97] F. Edgeworth. La teoria pura del monopolio. *Giornale degli Economisti*, 40:13–31, 1897.
- [EM81] Ilan Eshel and Uzi Motro. Kin selection and strong evolutionary stability of mutual help. *Theoretical Population Biology*, 19:420–433, 1981.
- [Esh83] I. Eshel. Evolutionary and continuous stability. *J. Theor. Biol.*, 103:99–111, 1983.
- [FK88] Drew Fudenberg and David Kreps. *Learning and Equilibrium in Games*. 1988.
- [FL95] Drew Fudenberg and David Levine. *Theory of Learning in Games*. unpublished book manuscript, 1995.
- [Fra85] Robert Frank. *Choosing the right pond: Human Behavior and the quest for status*. Oxford University Press, New York, 1985.
- [FT91] Drew Fudenberg and Jean Tirole. *Game theory*. MIT Press, Cambridge, Mass., 1991.
- [GP74] Victor Guillemin and Alan Pollack. *Differential Topology*. Prentice-Hall, Englewood Cliffs, NJ, 1974.
- [Hot29] Harold Hotelling. Stability in competition. *Economic Journal*, 39:41–57, 1929.
- [HS88] Josef Hofbauer and Karl Sigmund. *The Theory of Evolution and Dynamical Systems*. Cambridge University Press, New York, 1988.
- [Kau93] Stuart A. Kauffman. *The origins of order : self-organization and selection in evolution*. Oxford University Press, New York, 1993.
- [KMP96] Ken Kollman, John H. Miller, and Scott E. Page. Political institutions and sorting in a tiebout model. Unpublished manuscript, Cal Tech, Division of Humanities and Social Sciences, May 1996.
- [Lan76] Russell Lande. Natural selection and random genetic drift in phenotypic evolution. *Evolution*, 30:314–334, 1976.
- [Lan82] Russell Lande. A quantitative genetic theory of life history evolution. *Ecology*, 63:607–615, 1982.

- [Lax72] Peter Lax. The formation and decay of shock waves. *American Mathematical Monthly*, 66:227–241, 1972.
- [Lew92] Richard Lewontin. *Inside and outside: gene, environment and organism*. Clark University Press, Worcester, MA, 1992.
- [Log94] J. David Logan. *An Introduction to Nonlinear Partial Differential Equations*. Wiley-Interscience, New York, 1994.
- [Lom97] Bjorn Lomborg. Adaptive parties in a multipart multidimensional system with imperfect information. In *UCLA Computable Economics Conference*, Aarhus, Denmark, 1997. University of Aarhus Political Science Department.
- [LS94] Dan Levin and J. L. Smith. Equilibrium in auctions with entry. *American Economic Review*, 84:585–599, 1994.
- [LS96] Russell Lande and Susan Shannon. The role of genetic variation in adaptation and population persistence in a changing environment. *Evolution*, 103:434–437, 1996.
- [Mai92] George Mailath. Introduction: Symposium on evolutionary game theory. *Journal of Economic Theory*, 57:259–277, 1992.
- [MGM⁺95] J. A. J. Metz, S. A. H. Geritz, G Meszena, F.J.A.Jacobs, and J.S. van Heerwaarden. Adaptive dynamics: A geometrical study of the consequences of nearly faithful reproduction. Working Paper 95-99, IIASA, Laxenberg, Austria, September 1995.
- [Mil96] Igal Milchtaich. Congestion games with player-specific payoff functions. *Games and Economic Behavior*, 13:111–124, 1996.
- [MM95] Javier R. Movellan and James L. McClelland. Stochastic interactive processing, channel separability, and optimal perceptual inference: An examination of morton’s law. Technical Report PDP.CNS.95.4, University of California, Department of Cognitive Science, San Diego, CA, December 1995.
- [MS82] John Maynard-Smith. *Evolution and the theory of games*. Cambridge University Press, New York, 1982.
- [MS96] Dov Monderer and Lloyd Shapley. Potential games. *Games and Economic Behavior*, 14:124–143, 1996.
- [MSP73] John Maynard-Smith and G. R. Price. The logic of animal conflict. *Nature*, 246:15–18, 1973.
- [Nag95] Rosemarie Nagel. Unraveling in guessing games –an experimental study. *American Economic Review*, 95:1313–1326, 1995.
- [Rog95] Alan Rogers. Beating your neighbor to the berry patch. July 1995.

- [Ros73] Robert W. Rosenthal. A class of games possessing pure-strategy nash equilibria. *International Journal of Game Theory*, 26:65–67, 1973.
- [Rud73] Walter Rudin. *Functional Analysis*. McGraw-Hill, New York, 1973.
- [Sky86] Brian Skyrmes. Deliberational equilibrium. *Topoi*, 5:59–67, 1986.
- [Smo94] Joel Smoller. *Shock waves and reaction-diffusion equations*. Springer-Verlag, New York, second edition, 1994.
- [Son82] Hugo Sonnenschein. Price dynamics based on the adjustment of firms. *American Economic Review*, 72(5):1088–1096, 1982.
- [Tay95] Peter Taylor. The selection differential in quantitative genetics and ess models. August 1995.
- [Tir88] Jean Tirole. *The theory of industrial organization*. MIT Press, Cambridge, Mass., 1988.
- [TJ78] Peter Taylor and Leo Jonker. Evolutionarily stable strategies and game dynamics. *Mathematical Biosciences*, 40:145–156, 1978.
- [To95] Theodore To. Risk and evolution. *unpublished manuscript, St Andrews University Economics Dept*, 1995.
- [Veb99] Thorstein Veblen. *The theory of the leisure class*. MacMillan, London, 1899.
- [Wei94] Jonathan Weiner. *The beak of the finch : a story of evolution in our time*. Knopf - Random House, New York, 1994.
- [Wei95] Jorgen W. Weibull. *Evolutionary game theory*. MIT Press, 1995.
- [WFCB96] Donald Wittman, Daniel Friedman, Stephanie Crevier, and Aaron Braskin. Learning liability rules. *Journal of Legal Studies (in press)*, pages –, 1996.
- [Wri49] Sewall Wright. Adaptation and selection. In L. Jepson, G. G. Simpson, and E. Mayr, editors, *Genetics, Paleontology, and Evolution*. Princeton University Press, 1949.