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MONEY-MEDIATED DISEQUILIBRIUM PROCESSES IN A PURE EXCHANGE ECONOMY

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Studied here are dynamical processes of money-mediated exchange and price adjustment in a pure exchange economy of durables. The processes take place in continuous time and are specified by fields of cones which define admissible trajectories in price-allocation or allocation space. Under strong classical assumptions on preferences and fairly mild behavioral assumptions, stability theorems are proved, showing that our processes converge to price equilibria.

1. Introduction

The question of how one can model exchange and price adjustment in an economy outside of equilibrium was first systematically addressed in the non-tatonnement literature of the 1960's – see Arrow–Hahn (1972, ch. 13) for an excellent survey. Smale (1976) introduced several innovations in his approach to this question. For instance, he was able to relax significantly the assumptions of determinism (that the evolution of the economy is determined by a differential equation) and of agents' utility-maximizing behavior.

The work we present here draws heavily on Smale's methods. The other main source of inspiration is Clower (1969), who showed the importance of money-mediation in exchange. We introduce in section 2 the EP (Established Prices) process, featuring continuous-time money-mediated exchange and price adjustment, and briefly discuss institutional interpretations. A convergence theorem is proved in section 3, and the following section discusses the important deterministic case.

In section 5 we introduce the NP (Negotiable Prices) process. Here we drop the requirement that a unique price system always exists outside of equilibrium. We allow instead for a range of possible prices, the actual price

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used in a transaction being set by the parties involved. The final section mentions extensions of our basic results, some of which have been carried out and some of which remain conjectural.

Whenever possible, we use existing notation, mostly from Smale (1976), but the μ's of section 3 are adapted from Arrow–Hahn (1972) [see also Arrow–Hurwicz (1958)] and the T's from Grandmont (1976). Time derivatives are denoted by dots (e.g. $\dot{x}=dx/dt$), and partial derivatives are abbreviated by the subscripted ∂ (e.g., $\partial_k u(z)=\partial u(z^1,\dots,z^n)/\partial z^k$). Subscripts will generally index agents, and superscripts will index goods. Arguments of functions will be omitted when to do so makes for easier reading.

A fuller discussion of the material presented here can be found in Friedman (1977). Another important source of background information is Fisher (1976).

2. The EP process

Throughout this paper we consider a pure exchange economy of m agents and $l+1$ commodities, with fixed total resources $r\in R^{l+1}$, $r\gg 0$. Good 0, called *money*, will play a special role.

We assume that each agent's preference relation is *smooth* (C^2), that it satisfies differentiable versions of *strict monotonicity*[1] and *strict convexity*[2] and that it obeys a *strong boundary condition.* In terms of a representative utility function u_i for the ith agent's preference, these assumptions are:

(1) u_i is twice continuously differentiable on the strictly positive orthant $R^{l+1}_{++}=\{x_i\in R^{l+1}\,|\,x_i^k>0,\ k=0,\dots,l\}$.

(2) $\forall x_i\in R^{l+1}_{++},\ \partial_k u_i(x_i)>0,\ k=0,\dots,l.$

(3) $\forall x_i\in R^{l+1}_{++},\ D^2u_i(x_i)=(\partial^2_{jk}u_i)$, the matrix of second partial derivatives, is negative definite on the space of vectors tangent to the indifference surface through x_i, $\{v\in R^{l+1}\,|\,Du_i(x_i)v=0\}$.

(4) $\forall c\in R,\ u_i^{-1}(c)$ is closed in R^{l+1}, i.e., indifference surfaces do not meet the boundary of R^{l+1}_{++}.

Note that the vector of marginal utilities $(\partial_0 u_i(x_i),\dots,\partial_l u_i(x_i))$ is normal to the indifference surface through x_i and points in the direction of increasing utility. Using monotonicity in the 0th component, we may scale it so that the 0th component is 1, truncate, and get a function $g_i: R^{l+1}_{++}\to R^{l}_{++}$, whose kth component is $g_i^k(x_i)=\partial_k u_i(x_i)/\partial_0 u_i(x_i)$, the marginal rate of substitution of good k for money. These g_i are independent of the choice of u_i and will be used extensively.

[1]Note that we assume that money also has positive marginal utility, implying that it is a valuable good in its own right.

[2]This condition is often called 'strict quasi-concavity' in the literature.

We will take money as numeraire (this is justified by Proposition 2.3 in section 3), so *price space* is $\{1\} \times R^l_{++} \simeq R^l_{++}$. Thus a price vector $p \in R^l_{++}$ gives prices in terms of dollars. (Occasionally, p will mean $(1, p)$; the context should make the meaning clear.)

Let $r \in R^{l+1}_{++}$ be the given total endowment for the economy. Then *allocation space* is the set of attainable allocations

$$W = \left\{ x \in (R^{l+1}_{++})^m \,\middle|\, x = (x_1, \ldots, x_m),\ \sum_{i=1}^{m} x_i = r \right\}.$$

This says that we are in a pure exchange economy of durables, i.e., no production or consumption. The set of *Pareto optimal allocations* is denoted by $\theta \subset W$.

A *state* of this economy is an attainable allocation together with a price system, $(x, p) \in W \times R^l_{++}$.

A certain $(m-1)$-dimensional submanifold Λ of $W \times R^l_{++}$ is called the set of *price equilibria.* $(x, p) \in \Lambda$ means that $x \in \theta$ and p supports x, i.e., i's satisfaction is maximized at x_i on the set

$$\text{bud}_i(x_i, p) = \{ y_i \in R^{l+1}_{++} \,|\, p \cdot y_i \leqq p \cdot x_i \} \quad \text{for} \quad i = 1, \ldots, m.$$

From Smale (1974) we have the characterization by first-order conditions,

$$(x, p) \in \Lambda \Leftrightarrow g_i(x_i) = p, \qquad i = 1, \ldots, m.$$

We may now define a *process* as the specification of a state for each moment in some time interval, i.e., as a C^1 map,

$$\varphi : [a, b) \mapsto W \times R^l_{++},\ t \mapsto (x(t), p(t)), \qquad a < b \leqq \infty.$$

Before giving axioms for an EP process, we introduce two constructs related to money mediation.

We regard $g_i^k(x_i)$ as the dollar value that agent i imputes to the kth good if he has holdings x_i. Likewise, p^k is the dollar value that the market imputes to good k. Hence the difference $\mu_i^k(x_i, p^k) = g_i^k(x_i) - p^k$ is a measure of i's *desire to hold good k against money.* If $\mu_i^k > 0$ then evidently i would like to buy some k, while if $\mu_i^k < 0$ he would like to sell this good, at the given allocation and prices.

The other construct allows us to move from sales and purchases to changes in allocations. Let $\alpha_i \in R^l$ be a vector whose components represent the rates at which i is buying or selling the various goods. We adopt the convention that $\alpha_i^k > 0$ means that i is *buying* k, and $\alpha_i^k < 0$ means that i is *selling* k (at the rate $|\alpha_i^k|$ units per second, say). Then i's stock of money is changing at the rate $-p^k\alpha_i^k$ as he pays (or is paid) for good k. Hence the total effect on i's money stock corresponding to purchase/sales α_i is $-p\cdot\alpha_i = -\sum_{k=1}^{l} p^k\alpha_i^k$. We may summarize this in the $(l+1)\times l$ matrix

$$T(p)=\begin{pmatrix} -p \\ \hline I_l \end{pmatrix},$$

whose kth column is

$$T^k(p)=(-p^k,0,\ldots,0,\underset{\substack{\uparrow \\ k\text{th place}}}{1},0,\ldots,0)'.$$

$T(p)$ may also be thought of as the linear map $(-p;\, id): R^l \to R^1 \times R^l$. Then the total change in holdings corresponding to purchase/sales α_i is $T(p)\alpha_i$. $T(p)$ can also operate on α, taken as a matrix.

Now for the axioms. Let $\varphi:[a,b)\mapsto W\times R^l_{++},\, t\mapsto(x(t),p(t))$ be given. The *exchange axiom* is:

(E) For all $t\in[a,b)$;

(1) $\forall i, \dot{x}_i(t)=T(p(t))\alpha_i(t)$ for some $\alpha_i(t)\in R^l$.

(2) $\forall i, k, \alpha_i^k(t)\mu_i^k(x_i(t),p^k(t))>0$ or $\alpha_i^k(t)=0$.

(3) $\exists i, k$ such that $\alpha_i^k(t)\neq 0$ if possible subject to (2).

One can see that (E.2) says that agents buy only goods they value more than money, and sell only those they value less. (E.1), together with (E.2), tells us that transactions are money mediated; changes in allocation take place only through purchases and sales at market prices. Note that $x(t)\in W$, $\forall t$, $\Rightarrow \sum \dot{x}_i = 0 \Rightarrow \sum \alpha_i = 0$. Using this, one can see that $(E.3)$ means that some buying and selling will take place except when, for each k, the μ_i^k's all have the same sign.

The *price adjustment axiom* is:

(P) For some $\varepsilon>0$ and for each k:

$$\dot{p}^k(t)=\sum_{i=1}^{m} l_i^k\mu_i^k(x_i,p^k),$$

where $\varepsilon<l_i^k(t)<1/\varepsilon$, $\forall t\in[a,b)$, $i=1,\ldots,m$.

(P) says that the price of each good moves in response to the agents' demands for it against money, each agent always having at least some influence on the price of every good.

We may now define an *EP process* as a process which satisfies the exchange and price adjustment axioms.

EP processes may conveniently be described by the following cones: Let $(x,p)\in W\times R^l_{++}$. Then, the *purchase-sales cone* at (x,p) is

$$A'_{x,p}=\left\{\alpha\in(R^l)^m \,\middle|\, \alpha=(\alpha_1,\dots,\alpha_m);\ \sum_{i=1}^m \alpha_i=0;\ \alpha_i^k\mu_i^k(x_i,\ p^k)>0 \text{ or } \alpha_i^k=0\right\}.$$

The *exchange cone* is

$$\begin{aligned} A_{x,p}&=\{0\}\subset(R^{l+1})^m \quad \text{if} \quad A'_{x,p}=\{0\}\\ &=\{\bar{x}\in(R^{l+1})^m \,|\, \bar{x}=T(p)\alpha \quad \text{for some} \quad \alpha\in A'_{x,p}\}\backslash\{0\} \quad \text{otherwise.}\end{aligned}$$

The *price adjustment cone* is:

$$D_{x,p}=\bigcup_{\varepsilon>0} D_{x,p}(\varepsilon)$$

where

$$D_{x,p}(\varepsilon)=\left\{\bar{p}\in R^l \,\middle|\, \bar{p}^k=\sum_{i=1}^m l_i^k\mu_i^k(x_i,p^k),\ \varepsilon<l_i^k<1/\varepsilon\right\}.$$

Proposition 1

(a) $A'_{x,p}$, $A_{x,p}$, *and* $D_{x,p}$ *are all convex cones for every* (x,p).

Let $\varphi:[a,b)\to W\times R^l_{++}$ *be given. Then,*

(b) φ *satisfies Axiom* (E) *iff* $\dot{x}(t)\in A_{x(t),p(t)}$, $\forall t\in[a,b)$, *and*

(c) φ *satisfies Axiom* (P) *iff for some* $\varepsilon>0$, $\dot{p}(t)\in D_{x,p}(\varepsilon)$, $\forall t\in[a,b)$.

The proof is straightforward and has been omitted.

Proposition 2

(a) $(x,p)\in\Lambda$ *iff* $D_{x,p}=\{0\}$ *iff* $A_{x,p}\times D_{x,p}=\{0\}$, *and*

(b) $0\in D_{x,p}\neq\{0\}\Rightarrow A_{x,p}\neq\{0\}$.

Note: (a) says, among other things, that price change can always occur except at equilibrium and that only the trivial process [i.e., $\varphi(t)=\text{constant}$] is possible at equilibrium. (b) implies that if price changes are not forced, then, away from equilibrium, exchange must occur.

Proof. (a) follows from the following system of obvious implications:

$$(x,p)\in \Lambda \Leftrightarrow g_i(x_i)=p, \forall i \Leftrightarrow \mu_i^k(x_i,p^k)=0, \forall i, k \Leftrightarrow D_{x,p}=\{0\}$$
$$\Downarrow$$
$$A_{x,p}=\{0\}.$$

The hypothesis in (b) is: $\exists l_i^k$ such that $\sum_i l_i^k \mu_i^k=0$ but some $\mu_i^k \neq 0$. Let $\bar{\alpha}_i^k = l_i^k \mu_i^k$ and $\bar{x}_i = T(p)\bar{\alpha}_i$. Then $0 \neq (\bar{x}_1,\ldots,\bar{x}_m)\in A_{x,p}$. ∎

Example 1

Two agents (A and B) and three commodities (m (for money) plus 2 goods s and t). Total endowment $r=(1,1,1)$, and

$$u_A(m,s,t)=2\log m+\log s+\log t,$$
$$u_B(1-m,1-s,1-t)=\log(1-m)+\log(1-s)+\log(1-t).$$

Then we can easily compute the g's,

$$g_A(m,s,t)=(m/2s, m/2t),$$
$$g_B(1-m,1-s,1-t)=((1-m)/(1-s),(1-m)/(1-t)).$$

By the first-order conditions, we have $\theta=\{(m,s,t)|g_A \quad g_B\}$, and solving in terms of m we get $\theta=\{(m,m/(2-m),m/(2-m))|\in(0,1)\}$. We see that θ bows away from the main diagonal in the three-dimensional Edgeworth box towards the corner $(1,0,0)$. The corresponding equilibrium prices lie on the segment $[(1/2,\ 1/2),\ (1,\ 1)]$ in $p=(p^s,p^t)$-space (see fig. 2).

Now let us start an EP process at time $\tau=0$ at allocation $x_A(0)=(1/2,1/2,1/2)=x_B(0)$ and prices $p(0)=(1/2,1)$. We see that at this point,

$$\mu_A^s=1/2-1/2=0, \qquad \mu_A^t=1/2-1=-1/2,$$
$$\mu_B^s=1-1/2=1/2, \qquad \mu_B^t=1-1=0.$$

Evidently B would like to buy some s but A is not interested in selling it, and A would like to sell some t but B is not interested in buying it, so no exchange can take place according to Axiom (E). However, Axiom (P) implies that p^s must rise (at some rate between $\varepsilon/2$ and $1/2\varepsilon$) and p^t must fall. Thus prices will move in some direction in the price space cone depicted in fig. 1. Hence, after a short time τ, μ_A^s is negative and μ_B^s is positive, so by Axiom (E) we will have A selling some s to B and/or B selling some t to A. The cone in the Edgeworth box depicts the possible directions of allocation change at this point; it is flat and opens towards θ.

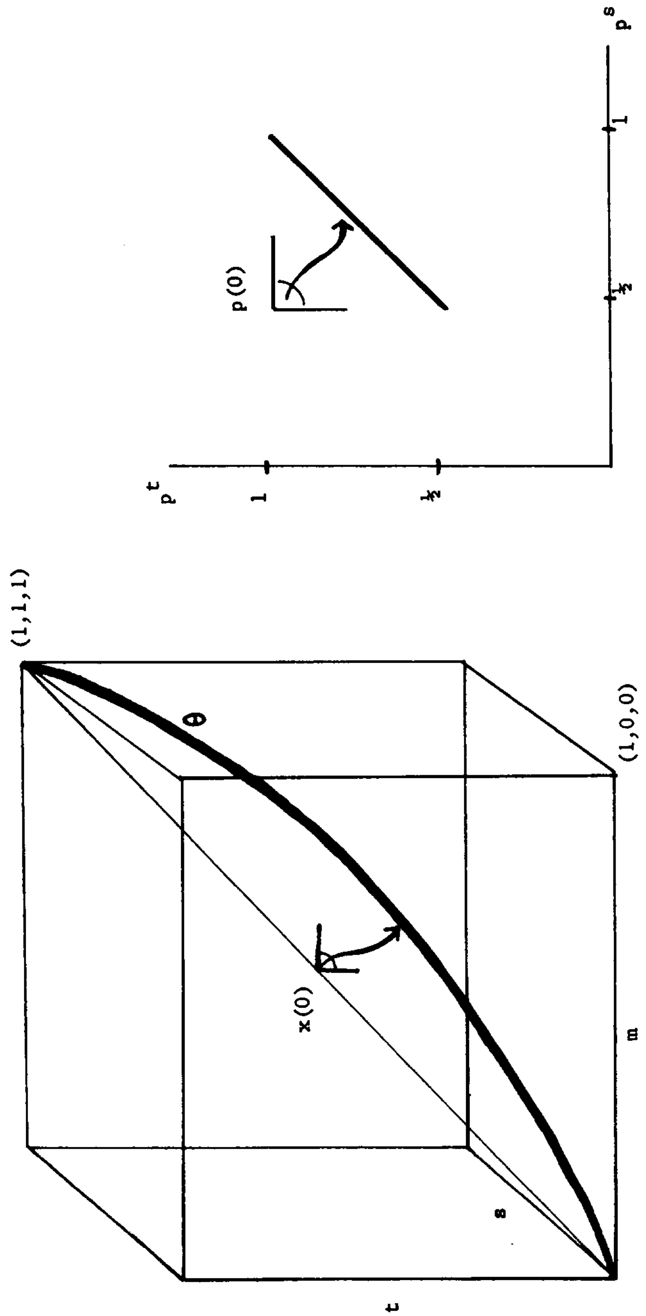

Fig. 1. Three-dimensional Edgeworth box and two-dimensional price space.

Let us now turn to the question of how EP processes may be interpreted institutionally. The requirement that at every time there is some particular price system $p(t)$ suggests the existence of trading posts to enforce the prices. We could suppose that there are l posts, one for each good other than money, at each of which agents can buy or sell the appropriate good for money at the going price. (E.2) indicates that the agents are more bargain hunters than utility-maximizers. Thus agent i wants to buy at post k if $\mu_i^k>0$, and may in this case receive good k at some rate $\alpha_i^k>0$ by paying money at rate $p^k\alpha_i^k$. Since we have made no provision for inventories, trader i will only be able to carry out these transactions if at that moment there is some other trader j who wants to sell ($\mu_j^k<0$). Balancing the intensities so $\sum_i \alpha_i^k(t)\equiv 0$ could be accomplished in several ways. The prospective buyers and sellers could be queued: once one buyer (seller) has bought (sold) all he wants, he is immediately replaced by another if there are any in line. Alternatively, all buyers and sellers could be active at once, with some rationing rule.

(E.3) says that agents are persistent and have some way of locating posts at which they can transact. We might suppose, for instance, that each post makes public announcements of unsatisfied demand or supply.

The boundary condition on the utility relations guarantees that agents will not attempt to buy anything if their money stock drops too low, or sell any good of which they have too little.

As for price adjustment mechanisms, Axiom (P) provides great flexibility. As long as transactions take place at post k, the price p^k can rise, fall, or (more plausibly) remain unchanged. When there is unsatisfied demand only (some $\mu_i^k>0$, no $\mu_i^k<0$), Axiom (P) ensures that p^k must rise. Likewise, p^k must fall if some agents want to sell but no one wants to buy. The structure of the μ's guarantees that prices will not collapse ($p^k\not\to 0$) or rise without bound – see Proposition 2.3 in section 3. No assumption is made on the speeds of price adjustment relative to the speeds of quantity adjustment.

Note that we need not assume that the process is deterministic. For instance, stochastic rationing or stochastic price changes may be employed, and the branching process of Arrow–Hahn (1972, p. 335ff) may also be adapted to our framework. Game theoretic strategic considerations could also be introduced as long as these strategies do not involve any sacrifices [which would violate Axiom (E.2)] or result in a situation in which all agents simultaneously hold back from all markets [violating (E.3)]. Such games are introduced in a slightly different context (no prices, for instance) in Tulkens–Zamir (1976).

3. Convergence of the EP process

We first note that EP processes have an Edgeworth property that agents' satisfaction is non-decreasing.

Proposition 1. *Let $\varphi: t \to (x(t), p(t) \in W \times R^l_{++}$ satisfy Axiom* (E). *Then each $u_i(x_i(t))$ is non-decreasing in t. If $A_{x(t),\,p(t)} \neq \{0\}$, then $V(t) = \sum_{i=1}^m u_i(x_i(t))$ is strictly increasing at t.*

Proof

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} u_i(x_i(t)) &= \operatorname{grad} u_i(x_i(t)) \cdot \dot{x}_i(t) \\
&= \partial_0 u_i(x_i)(1, g_i(x_i)) \cdot (T(p)\alpha_i) \\
&= \partial_0 u_i(x_i) \left[-\sum_{k=1} p^k \alpha_i^k + \sum_{k=1} g_i^k(x_i)\alpha_i^k \right] \\
&= \partial_0 u_i(x_i) \sum_{k=1}^{l} \alpha_i^k \mu_i^k(x_i, p^k).
\end{aligned}$$

Each term in the last sum is non-negative by (E.2) and $\partial_0 \mu_i(x_i) > 0$ by monotonicity, so the first part of the proposition follows. If $A_{x,p} \neq \{0\}$ then one of the terms is positive by (E.3), yielding the second part of the proposition. ▌

Our main interest is in the behavior of $\varphi(t)$ as $t \to b$. By $\lim \varphi(t)$ we mean the set of all limit points (or adherent points)

$$\bigcap_{a \leq t < b} \overline{\varphi[t, b)} = \{(x; p) \in W \times R^l_{++} \mid \exists \text{ an increasing sequence } \{t_n\} \text{ s.t. } t_n \to b \text{ and } \varphi(t_n) \to (x, p) \text{ as } n \to \infty\}.$$

If this set is a singleton $\{(x, p)\}$ we write $(x, p) = \lim_{t \to b} \varphi(t)$.

Proposition 2. *Let $\varphi: [a, b) \to W \times R^l_{++}$ be an EP process. Then*

(1) $\lim \varphi(t)$ *is a non-empty, compact, connected subset of* $W \times R^l_{++}$;

(2) $\lim x(t) \subset u^{-1}(c)$ *for some* $c \in R^m$; *and*

(3) $p^k(t)$ *is bounded and bounded away from* 0, $k = 1, \ldots, l$.

Most of the proof is analogous to arguments in Smale (1976); see Friedman (1977) for details.

With Proposition 2 in hand we may consider the potential obstacles to reaching price equilibrium. The first is of a trivial sort: the process may stop short of equilibrium just becuase we closed down the market too soon. We say, following Smale (1976), that φ is *incomplete* if either:

(a) $\lim_{t\to b}\varphi(t)=(x,p)$ but non-trivial EP processes[3] may begin at (x,p), *or*

(b) $\lim_{t\to b}x(t)=x$ but non-trivial exchange[3] is possible in accordance with Axiom (E) starting at (x,p) for *every* $p\in\lim p(t)$.

Then φ is *complete* if it is not incomplete. Note that complete processes are not assumed to converge; they are just not allowed to converge prematurely.

A more serious obstacle to reaching equilibrium is that the process may not converge. To ensure convergence of $x(t)$, the sufficiency of two mild conditions is demonstrated in the next proposition. Other sufficient conditions and discussion can be found in Friedman (1977). Whether any of these conditions are necessary remains an open question.

Proposition 3. Let φ be an EP process and let $V(t)=\sum_i u_i(x_i(t))$. Then $x(t)$ converges as $t\to b$ if either of the following conditions is satisfied:

(1) $\dot V(t)\geqq\delta\|\dot x(t)\|$ *for some* $\delta>0$ *and all* $t\in[a,b)$, *or*

(2) $\lim x(t)\cap\theta\neq\phi$.

Remarks. We say that φ is *responsive to transactions costs* if (1) holds outside of any neighbourhood of θ. The meaning is that agents will not make sizable transactions for an arbitrarily small (all over) gain in satisfaction. This could be enforced if the trading posts levy a small sales tax, or if there are essentially any kind of underlying transaction costs, either fixed or marginal. In keeping with the generality of this presentation, we will not directly incorporate such transactions costs or taxes into our processes.

Proof. By Proposition 2.2, there is some $c\in R^m$ s.t. $\lim x(t)\subset u^{-1}(c)$. By Proposition 1, $V(t)\nearrow d=\sum c_i$ as $t\to b$. Therefore if (1) holds, arclength $(x(t))$ $\equiv\int_a^b\|\dot x(t)\|\,\mathrm{d}t\leqq(1/\delta)\int_a^b\dot V(t)\,\mathrm{d}t=(1/\delta)(d-V(a))<\infty$. Hence $x(t)$ converges.

For (2), let $y\in\lim x(t)\cap\theta$ and $z\in\lim x(t)$. By Proposition 2.2 again, $u_i(y_i)$ $=c_i=u_i(z_i)$, $i=1,\dots,m$, so z must also be Pareto Optimal. But u is $1:1$ on θ, so $y=z$. Hence $\lim x(t)$ is a singleton, i.e., $x(t)$ converges. ▮

We turn now to the question of the convergence of $p(t)$. If $x(t)\to x$ and $x\notin\theta$ then there will be some good k and agents i, j for which $g_i^k(x_i)<g_j^k(x_j)$. In such a case, Axiom (P) will allow $p^k(t)$ to bounce around in $(g_i^k(x_i),\, g_j^k(x_j))$, so we see that there is no reason to believe in general that $p(t)$ will converge, even if $x(t)$ does. However, exchange will always be possible in this case, as we now see.

[3]For the moment, this only means that the cone at (x,p) is non-trivial. Proposition 1 in section 4 shows that this is equivalent to the existence of a non-constant φ with $\varphi(a)=(x,p)$.

Proposition 4. If φ is an EP process such that $x(t) \to x$ as $t \to b$, and if $p \in \lim p(t)$, then either $A_{x,p} \neq \{0\}$ or $p(t)$ actually converges to p.

The proof is a bit technical and is omitted. The idea is that limit prices, if not unique, must lie within the price box B_x (defined below in section 5), implying that at allocation x agents differ in their valuation of some good and hence the exchange cone is non-trivial.

We are now ready to state and prove the main result of this section.

Theorem. Every complete EP process $\varphi:[a,b) \to W \times R^l_{++}$ which is responsive to transactions costs converges to a price equilibrium as $t \to b$.

Proof. Proposition 3.1 implies that $x(t)$ converges, say to x. If $p(t)$ also converges, say to p, and $(x,p) \notin \Lambda$ we see by Proposition 2 in section 2 that $A_{x,p} \times D_{x,p} \neq \{0\}$ so φ is not complete. If $p(t)$ does not converge, and $p \in \lim p(t)$, then by Proposition 4, $A_{x,p} \neq \{0\}$, so again φ is not complete. Hence φ converges to some $(x,p) \in \Lambda$. ∎

Evidently, the completeness condition weighs heavily in the proof. Although this condition is reasonable in its context, there are times when one would prefer a condition on the process itself rather than on its limit. To this end, we introduce the notion of responsiveness to desires.

Let

$$J^k_{\underset{(s)}{b}}(x,p^k) = \{j \mid \mu^k_j(x_j,p^k) \underset{(<)}{>} 0\}$$

be the 'buyers' (resp. 'sellers') of good k at the given allocation and price. Let

$$\mu^k_b = \sum_{i \in J^k_b} \mu^k_i, \qquad \mu^k_s = \sum_{i \in J^k_s} \mu^k_i,$$

and let $C^k = \min\{\mu^k_b, -\mu^k_s\}$, so C^k is a measure of the strength of the weaker side of the market, which we can take to be the limiting factor on commerce. We say that φ is *responsive to desires* if for some $\delta > 0$. $\|\dot{x}^k(t)\| \geqq \delta C^k(t), \forall t \in [a,b)$.

Proposition 5. If the EP process $\varphi:[a,\infty) \to W \times R^l_{++}$ is responsive to desires, then it is complete. If φ is also responsive to transaction costs, then it converges to a price equilibrium.

Proof. If $x(t)$ does not converge, then φ is certainly complete. Suppose $x(t) \to x$ as $t \to \infty$. Then $\liminf_{t\to\infty} \|\dot{x}^k(t)\| = 0$, so by the responsiveness condition,

$\liminf_{t\to\infty} C^k(t)=0$ for all k. Hence, $A_{x,p}=\{0\}$ for some $p\in\lim p(t)$. By Proposition 4, this implies that $p(t)\to p$. Therefore $0\in D_{x,p}$, and in fact, by Proposition 2 in section 2, $D_{x,p}=\{0\}$. $\therefore$ φ is complete. The rest follows immediately from the theorem. ▮

4. Deterministic EP process

We say that a process φ is *deterministic* if the parameters $\alpha(t)$ and $l(t)$ in Axioms (E) and (P) depend only on the present state $\varphi(t)=(x(t),p(t))$. If $\alpha(x,p)$ and $l(x,p)$ are defined on all of $W\times R^l_{++}$ and not just on imφ, then we have a *globally defined* deterministic model. Setting $f(x,p)=T(p)\alpha(x,p)$ and $g(x,p)=\sum_i l_i(x,p)\mu_i(x_i,p)$, we see that such a model has the form

$$\dot{x}=f(x,p), \qquad \dot{p}=g(x,p). \tag{1}$$

If we pick some initial conditions $(x(0),p(0))\in W\times R^l_{++}$, then the solution to (1) yields an EP process $\varphi:[0,\infty)\to W\times R^l_{++}$.

Example 1. Full information: Suppose that at each realized state of the economy, the agents communicate to the trading post managers their desires μ, or equivalently, they give their preference relations at the outset. Let the managers regulate exchange according to relative desires, subject to market clearing conditions, and move prices according to aggregate desires. The simplest such model can be expressed in terms of $\alpha(x,p)$ and $l(x,p)$ as follows:

$$l_i^k\equiv 1,$$

$$\alpha_i^k=\mu_i^k(x_i,p^k)\left|\sum_{j\in J_i^k}\mu_j^k\right|,$$

where

$$J_i^k(x,p)=\{j\,|\,\mu_i^k\mu_j^k<0\}$$

is the 'other side of the market', $i=1,\dots,m$, $k=1,\dots,l$.

One notes that this kind of proportional rationing provides an incentive for dishonesty, which is eliminated in the next example.

Example 2. Minimum information: Suppose at each state the agents indicate to the managers only a desire to buy or sell or neither – i's message to k is $c_i^k(x_i,p^k)=\operatorname{sgn}\mu_i^k=1,\ -1$, or 0. Let each manager fix the aggregate speed of exchange at unity and hold prices constant if trade is possible. Otherwise, let him move prices in the appropriate direction. We specify price adjustment in terms of $g(x,p)$ by

$$\dot{p}^k = [g(x,p)]^k = \sum_i c_i^k \quad \text{if} \quad J_b^k = \phi \quad \text{or} \quad J_s^k = \phi,$$
$$= 0 \quad \text{otherwise,}$$

where $J_b^k = \{j \mid c_j^k = 1\}$ and $J_s^k = \{j \mid c_j^k = -1\}$ are the 'buyers' and 'sellers'. Specify exchange by

$$\alpha_i^k = 0 \quad \text{if} \quad J_b^k = \phi \quad \text{or} \quad J_s^k = \phi,$$
$$= c_i^k / \# J_a^k, \qquad a = \begin{cases} b & \text{if} \quad c_i^k = 1 \quad \text{or} \quad 0 \\ s & \text{if} \quad c_i^k = -1. \end{cases}$$

This model yields processes which may have discontinuities in the first derivative. However, by slight extensions of our definitions, they can be regarded as EP processes – see Friedman (1977).

Example 3. General, a priori: One may pick any finite set of states and arbitrarily specify admissible vectors of exchange and price adjustment at these points. Proposition 1 below shows that one may then interpolate globally defined functions $f(x,p)$ and $g(x,p)$, generating smooth EP processes. This method is due to Smale (1976).

Proposition 1. Given a finite number of states $({}_jx, p_j) \in W \times R^l_{++}$ and corresponding admissible vectors $(f_j, g_j) \in A_{{}_jx,p_j} \times D_{{}_jx,p_j}$, $j=1,\dots,n$, there is a continuous vector field $f \times g$ on $W \times R^l_{++}$ such that

(1) $f({}_jx, p_j) = f_j$ and $g({}_jx, p_j) = g_j$, $j = 1,\dots,n$;

(2) $\exists \varepsilon > 0$ s.t. $\forall (x,p) \in W \times R^l_{++}$, $f(x,p) \in A_{x,p}$ and $g(x,p) \in D_{x,p}(\varepsilon)$; and

(4) g is C^1 and f is C^1 off $f^{-1}(0)$.

The proof, involving local constant extensions pieced together by a partition of unity, is straightforward but technical and is omitted.

One immediate consequence of this proposition is the existence of EP processes beginning at any intial state.

The hypothesis of determinism is rather strong, and eliminates the need for additional conditions to guarantee convergence, as we now show.

Theorem. Every deterministic EP process $\varphi: [0,\infty) \to W \times R^l_{++}$ converges to a price equilibrium as $t \to \infty$.

Proof. The methods of the previous proposition allow us to assume without loss of generality that φ is a solution to some differential equation of the form (1) above, with with initial conditions $({}_0x, p_0)$. By Proposition 2 in section 3 $\lim\varphi(t)$ is non-empty.

Claim. For every $(x,p)\in\lim\varphi(t)$, $f(x,p)=0$ and $A_{x,p}=\{0\}$.

Let $\Phi_t(x,p)=(F_t(x,p), G_t(x,p))$ be the solution to (1) from initial conditions (x,p). [Thus, $(x(t), p(t))=\varphi(t)=\Phi_t({}_0x, p_0)$.] Suppose $f(x,p)\neq 0$. Then given $\varepsilon>0$ sufficiently small $\exists T>0$ s.t.

$$V(F_t(x,p))>V(x)+\varepsilon, \qquad \forall t>T, \tag{2}$$

by Proposition 1 in section 3. But F_t is continuous in (x,p) and V is continuous, so (2) will hold for any (x^*,p^*) sufficiently close to (x,p). If $(x,p)\in\lim\varphi(t)$, then $(x(\tau), p(\tau))$ gets arbitrarily close to (x,p). Now $F_t(x(\tau), p(\tau)) = F_{t+\tau}({}_0x, p_0)=x(t+\tau)$, so (2) implies that $V(x(t+\tau))>V(x)$, contradicting the fact that $x\in\lim x(t)$ by Proposition 1 in section 3. This proves that $f=0$ on $\lim\varphi(t)$; the last part of the claim follows from the basic fact that $0\in A_{x,p}\Rightarrow A_{x,p}=\{0\}$.

Now the well-known invariance property of solutions to DE's tells us that $\Phi_t(x,p)\subset\lim\varphi(t)$, $\forall(x,p)\in\lim\varphi(t)$. Pick $(x,p)\in\lim\varphi(t)$ and let $(\bar{x},\bar{p})\in\lim\Phi_t(x,p)$. By the claim above we have $F_t(x,p)\to\bar{x}$ [in fact $F_t(x,p)\equiv\bar{x}$], so we may apply Proposition 4 in section 3 (noting $A_{\bar{x},\bar{p}}=\{0\}$) to conclude that $G_t(x,p)\to\bar{p}$ also. As a consequence, $0\in D_{\bar{x},\bar{p}}$. By Proposition 2 in section 2 it follows that $D_{\bar{x},\bar{p}}=\{0\}$ and $(\bar{x},\bar{p})\in\Lambda$. In particular $\bar{x}\in\theta$ so by Proposition 3 in section 3, we find that actually $x(t)\to\bar{x}$. Using Proposition 4 in section 3 again we see that $p(t)\to\bar{p}$. ▮

Remarks. Schecter (1975) shows that arclength is finite for any convergent process, so we may reparametrize to get convergence in finite time, if we wish.

We may use the methods of Proposition 1 to extend a vector field defined on any closed set, so one can actually show that any convergent EP process may be *a posteriori* regarded as deterministic by extending the vectorfield from $\mathrm{Im}\,\varphi$ to all of $W\times R^l_{++}$.

5. The EP process

In many markets of interest, there seems to be a *range* of prices, rather than a uniquely determined price, at which goods may be bought and sold, the range being quite wide when the market is far from equilibrium. The measure μ of agents' desires to buy and sell, introduced in section 2, suggests a way to model this sort of market. Suppose for some good k and two agents

i and j, we have $g_j^k(x_j) < g_i^k(x_i)$. If they can agree on a price p^k between the two marginal valuations of k, then they will both benefit if i buys some k from j since in this case $\mu_i^k(x_i, p^k) > 0 > \mu_j^k(x_j, p^k)$. Hence we define the open interval $I_{ij}^k(x) = (g_j^k(x_j), g_i^k(x_i))$ as the *range of mutually beneficial prices* (for i to buy k from j, given x).

Some more useful notation: Let

$$I^k(x) = \overline{\bigcup_{i,j} I_{ij}^k(x)} = [\underline{p}^k(x), \bar{p}^k(x)],$$

where

$$\bar{p}^k(x) = \max_i \{g_i^k(x_i)\} \quad \text{and} \quad \underline{p}^k(x) = \min_i \{g_i^k(x_i)\},$$

and let

$$B_x = I^1(x) \times I^2(x) \times \ldots \times I^l(x) \subset R_{++}^l.$$

Finally, let $B_x^0 = B_x \backslash \{\text{corners}\}$. (By convention, a singleton has no corners.)

In terms of the EP process, B_x can be interpreted as the box in price space whose relative interior is the set of price systems sustainable under Axiom (P). For present purposes, though, we will call B_x the *box of mutually beneficial prices:* $p = (p^1, \ldots, p^l) \in \operatorname{int} B_x$ iff for each k there are agents who can exchange some good k to mutual benefit.

We are now prepared to introduce the NP process more formally. Let preferences, allocation space, price space, g, μ and $T(p)$ be as defined in section 2. By a NP process we mean a C^1 map x: $[a, b) \to W$ satisfying the following exchange axiom:

(E′) For each $t \in [a, b)$:

(1) $\forall i$, $\dot{x}_i(t) = \sum_{j=1}^m T(p_{ij})\alpha_{ij}$, for some $p_{ij} = p_{ij}(t) \in R_{++}^l$, $\alpha_{ij} = \alpha_{ij}(t) \in R^l$.

(2) $\forall i, j$, $\alpha_{ij} + \alpha_{ji} = 0$.

(3) $\forall i, j, k$, $\alpha_{ij}^k > 0$ only if $p_{ij}^k = p_{ji}^k \in I_{ij}^k(x(t))$.

(4) $\exists i, j$, such that $\alpha_{ij} \neq 0$ if possible subject to (2) and (3).

(E′.1) says that each agent's infinitesimal trade consists of combinations of bilateral money mediated transactions with the other agents. (E′.2) makes this bilterality explicit. (E′.3) says that i buys k from j only if they agree on a mutually beneficial price. Note that (E′.3) implies that $\alpha_{ij}^k \mu_i^k(x_i, p_{ij}^k) > 0$ unless $\alpha_{ij}^k = 0$, so it plays a role similar to (E.2). Finally, (E′.4), like (E.3), says that at least some buying and selling will take place if possible.

This axiom can be expressed in terms of an exchange cone. Let

$$C_x=\left\{\bar{x}\in(R^{l+1})^m \,|\, \bar{x}_i=\sum_{j=1}^{m} T(p_{ij})\alpha_{ij};\right.$$

$$\left.\alpha_{ij}+\alpha_{ji}=0;\ \alpha_{ij}^k>0\Rightarrow p_{ij}^k=p_{ji}^k\in I_{ij}^l(x)\right\}.$$

Proposition 1

(a) C_x *is a convex cone for each* $x\in W$,

(b) $C_x=\{0\}$ *iff* $B_x=\{p(x)\}$, *and*

(c) $x(t)$ *satisfies* (E′) *iff* $\dot{x}(t)\in C_{x(t)}$ *and* $\dot{x}(t)\neq 0$ *if* $C_x\neq\{0\}$.

The straightforward proof is omitted.

This model admits many economic interpretations. Certainly any economy which satisfies the EP process axioms will satisfy these: the α_i's of (E) can be decomposed into α_{ij}'s, and we may set $p_{ij}=p(t)$, $\forall i,j$, in which case (E)$\Rightarrow$(E′).

But the NP process is more general. We need not hypothesize trading posts, since prices are not enforced. We can imagine a setting where agents meet in groups of two or more, bargain with each other, agree on prices, and then buy and sell various goods.

The market could also be somewhat more organized. For instance, agents could be thought of as dealers of the Fisher (1972) sort, who advertise prices $\tilde{p}_i$ at which they are willing to sell so ($\tilde{p}_i^k>g_i^k(x_i)$), and wait for willing buyers to visit them. Alternatively, agents could announce prices that they are prepared to pay ($\tilde{\tilde{p}}_i^k<g_i^k(x)$) and wait for salesmen to drop by. Or perhaps a combination of the above, via a system of classified ads. Whatever the realization, (E′.4) requires that the market be efficient enough for at least one deal to be consumated at all times in which mutually beneficial deals are possible.

Despite this great flexibility of prices, it is not hard to construct examples which show that NP processes are more restrictive than general barter processes. They do retain the important Edgeworth property, however:

Proposition 2. If $x(t)$ *satisfies* (E′), *then* $u_i(x_i(t))$ *is non-decreasing for each i, and* $V(t)=\sum_i u_i(x_i(t))$ *is strictly increasing for all t such that* $C_{x(t)}\neq\{0\}$.

The proof is similar to that of Proposition 1 in section 3.

The following two conditions will prove useful:

Condition T. Given any open neighbourhood N of θ ($\theta \subset N \subset W$), there is some $\delta > 0$ such that $\dot{V}(t) \geqq \delta \|\dot{x}(t)\|$ for all $t \in [a, b)$ such that $x(t) \notin N$.

In other words, there are 'transactions costs', at least away from θ. Other mild conditions guaranteeing the convergence of $x(t)$ undoubtedly could be found.

Condition C. Either $x(t)$ does not converge, or $x(t) \to x$ as $t \to b$, and the only NP process starting at x is trivial.

As with the EP process, this completeness condition could be replaced by a responsiveness to desires condition, such as $\|\dot{x}^k(t)\| > \delta(\bar{p}^k(t) - p^k(t))$, for the case $b = \infty$.

Determinism seems less relevant to the spirit of the NP process so we do not discuss it explicitly, but apparently a hypothesis of determinism could still replace Conditions C and T.

Proposition 3. If $x: [a, b) \to W$ satisfies Axiom (E′) *and Condition* T, *then* $\lim_{t \to b} x(t)$ *exists and is in W.*

The proof is exactly like that of Proposition 3.1 in section 3.

The main result of this section is the following:

Theorem. If the NP process x: $[a, b) \to W$ satisfies Conditions T *and* C, *then* $\lim x(t)$ *exists, $B_{x(b)}$ is a singleton $\{p(x(b))\}$, and $(x(b), p(b)) \in \Lambda$. Such processes exist starting from any allocation $x_0 \in W$.*

The general idea behind the theorem is that $x(t)$ converges to a Pareto Optimal allocation while $B_{x(t)}$ shrinks to a point.

One part of the theorem is immediate:

Proposition 4. B_x is a singleton $\{p\}$ iff $(x, p) \in \Lambda$.

Proof

$$\begin{aligned} B_x = \{p\} &\Leftrightarrow \bar{p}^k = \underline{p}^k(x) = p^k, \forall k \\ &\Leftrightarrow g_i^k(x) = p^k, \forall i, k \\ &\Leftrightarrow (x, p) \in \Lambda. \quad \blacksquare \end{aligned}$$

We now proceed with the proof of the theorem.

Let x: $[a,b)\to W$ satisfy Axiom (E′), and Conditions T and C. By Proposition 3, $\lim_{t\to b} x(t)=x(b)\in W$. Suppose $B_{x(b)}$ is not a singleton. Then $C_{x(b)}\neq\{0\}$ by Proposition 1.b, so pick $f_0\neq 0$ in $C_{x(b)}$, $p\in B_{x(b)}$, and apply Proposition 1 in section 4 to get a vector field $f(x,p)$ on W extending f_0. For x near $x(b)$, $f(x,p)\in C_x$ so the solution to the differential equation $\dot{x}=f(x,p)$ with initial condition $x(b)$ is a non-trivial extension of x, violating C. Hence $B_{x(b)}$ is a singleton, so $(x(b),p(b))\in\Lambda$ by Proposition 4.

The existence part of the theorem follows from the existence of EP processes, since an EP process always gives rise to an NP process. ▮

6. Extensions

Friedman (1977) introduces discrete-time versions of both EP and NP processes, and proves convergence theorems analagous to Theorems 1, 2 and 3 above. This paper also shows that, at the cost of some technical complication, the strong boundary and monotonicity conditions of section 2 may be greatly relaxed. Wan (1977) reports that the convexity condition may also be substantially weakened.

At present, the question of accessibility remains open. For Edgeworth processes (barter, no prices) Schecter (1975) showed that given any initial state y and any $z\in\Lambda$ with z preferred to y by all agents, there is an admissible trade curve $x(t)$ with $x(0)=y$ and $x(1)=z$. His methods can easily be extended to give the same result in a 2-person economy for NP processes, but the question is open in general for EP and NP processes, as well as for Smale's Exchange, Price Adjustment processes.

A very interesting problem, suggested by L. Hurwicz, is to derive an EP-type process with fiat money, transactions costs and possibly completeness from a basic model which has some kind of transactions costs.

Perhaps the largest unanswered question, regarded by many as the fundamental problem of Equilibrium Theory, is to find reasonable dynamics for an economy with perishable goods, consumption and production. For this undertaking, the processes of this paper are inadequate, but we hope they will prove helpful.

References

Arrow, K. and F.H. Hahn, 1972, General competitive analysis (Holden-Day, San Francisco, CA).

Arrow, K. and L. Hurwicz, 1958, On the stability of the competitive equilibrium I, Econometrica 26, no. 4, 522–552.

Clower, R.W., 1969, Readings in monetary theory (Penguin, London).

Fisher, F., 1972, On price adjustment without an auctioneer, Review of Economic Studies 39, Jan., 1–16.

Fisher, F., 1976, The stability of general equilibrium: Results and problems, in: Artis and Nobay, eds., Essays in economic analysis, Proceedings of the Association of University Teachers of Economics Annual Conference, Sheffield, 1975 (Cambridge University Press, Cambridge) 3–29.

Friedman, D., 1977, Disequilibrium processes in a monetary economy of pure exchange, Ph.D. thesis (University of California, Santa Cruz, CA).

Grandmont, J.M., 1976, Temporary general equilibrium theory, CEPREMAP discussion paper, Jan., Econometrica, forthcoming.

Schecter, S., 1975, Smooth pareto economic systems with natural boundary conditions, Ph.D. thesis (University of California, Berkeley, CA).

Smale, S., 1976, Exchange processes with price adjustment, Journal of Mathematical Economics 3, no. 3, 211–226.

Tulkens, H. and S. Zamir, 1976, Local games in dynamic exchange processes, CORE discussion paper no. 7606, April (CORE, Louvain).

Wan, Y.H., 1977, Personal communication, Feb. (Department of Mathematics, SUNY, Buffalo, NY).