



Evolutionary dynamics over continuous action spaces for population games that arise from symmetric two-player games [☆]

Daniel Friedman ^{a,*}, Daniel N. Ostrov ^b

^a *Economics Department, University of California, Santa Cruz, CA 95064, United States*

^b *Department of Mathematics and Computer Science, Santa Clara University, 500 El Camino Real,
Santa Clara, CA 95053-0290, United States*

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Abstract

Any absolutely continuous, piecewise smooth, symmetric two-player game can be extended to define a population game in which each player interacts with a large representative subset of the entire population. Assuming that players respond to the payoff gradient over a continuous action space, we obtain nonlinear integro-partial differential equations that are numerically tractable and sometimes analytically tractable. Economic applications include oligopoly, growth theory, and financial bubbles and crashes.

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* Corresponding author.

E-mail addresses: dan@ucsc.edu (D. Friedman), dostrov@scu.edu (D.N. Ostrov).

1. Introduction

Classic game theory is formulated in terms of two or more distinct individuals who interact strategically. Population games, by contrast, consider anonymous interactions of large numbers of strategically identical players. Classic games and population games are nevertheless complementary and mutually illuminating, as noted in John Nash's 1950 dissertation, by John Maynard Smith [21], and by numerous more recent authors, e.g., Sandholm [28].

The current paper examines an important connection between the two theoretical traditions: single population games that arise from a symmetric two player game with a continuous action space. To generate a player's payoff in the population game, one takes the weighted average payoff that player would receive in the two player game, with weights given by the current population distribution of actions.

Our contribution is to derive and analyze natural dynamics for such games. The idea is that each player adjusts his action continuously, seeking higher payoff by following the gradient. Of course, such adjustments change the action distribution and thus the payoff gradient. The dynamic interplay between the action distribution and the payoff gradient is described by an integro-partial differential equation, where the integral is over the action space. Solutions are time paths of action distributions. Some specifications of the underlying games yield solutions that converge smoothly to a singular distribution in which all players choose the same action, while other specifications yield solutions with moving singularities and/or persistently diffuse behavior. Applications span the social sciences as well as evolutionary biology.

Section 2 shows how to obtain a population game with payoff ϕ from an arbitrary symmetric two-player game g over a continuous action space A . It then derives the partial differential equation describing the how the population distribution changes under gradient dynamics. The section also presents a simple family of examples that illustrates the diversity of possible behaviors arising from the PDE.

Section 3 explores games that yield linear gradient dynamics PDEs. The first theorem establishes that solutions will be "classical" — that is, smooth — for a broad class of underlying games g , including those that lead to linear PDEs, when the action space A has no boundary points. It then introduces the method of characteristics from partial differential equation theory, and shows how to apply this method to understand the solution of linear gradient dynamics PDEs, including how population mass can pile up at a boundary point of the action space A . The section then works out dynamics for several notable examples in social science: Cournot oligopoly, Keynes' beauty contest, and growth theory. It concludes by fully characterizing population behavior when players' choices consist of strategy mixtures for an underlying 2×2 game.

Section 4 explores games that yield nonlinear gradient dynamics PDEs. The solution in these cases can, in finite time, develop singularities — moving kinks or discontinuities, known in the mathematical literature as "shock waves". The PDE, of course, requires re-interpretation in such non-differentiable cases, but we show how to extend the method of characteristics to do so. The section includes applications to Bertrand competition and to financial bubbles and crashes. Section 5 offers a brief concluding discussion. Appendix A collects the more technical proofs. Finally, Appendix B describes the numerical techniques used to compute the PDE solutions shown in the figures throughout this paper. These techniques are standard in the study of numerical PDE.

Our work connects several strands of existing literature. One strand considers evolutionary games with continuous strategy spaces, and uses the topology of the strategy space to define convergence but not to describe individual adjustment behavior (e.g., Oechssler and Riedel [23];

Cressman and Hofbauer [8]; Ruijgrok and Ruijgrok [27]; and Hofbauer et al. [16]). An older strand of literature (e.g., Simon [29]; Arrow and Hurwicz [2]; Luenberger [20]; recent examples include Hart and Mas-Colell [15], and Anderson et al. [1]) uses gradient dynamics to model individual adjustment in n -player games or games against nature, but does not consider population games.

The current paper builds on Friedman and Ostrov [13], which includes a brief survey of these and other strands of existing literature in economics, biology and mathematics. That paper motivates gradient dynamics, and proves that in an appropriate sense they represent optimal response to quadratic adjustment cost. The first part of that paper focuses on a class of “local” population game payoff functions that cannot typically be obtained from two-player games, and shows how known results from the mathematical physics literature imply that unique solutions exist for the resulting PDEs. The rest of that paper focuses on convergence issues, in particular for population games with potential functions, and connects steady state distributions to local Nash equilibria. The focus of the present paper is different. When there is an underlying symmetric two player game, we show how to obtain explicit solutions in important cases, and how to obtain numerical solutions more generally. We prove three new theorems on classical solutions, and demonstrate solution techniques (both classical and “shock wave”) for a variety of economic applications.

2. Extending two-player games

In this paper we study a class of games with a single population of strategically identical players. Each player has the same continuous action space A , a closed interval of real numbers. In economic applications, $x \in A$ might represent price, or output quantity, or location, or product quality. In biological applications, it might represent a continuous trait such as beak size or migration date. Our analysis will focus on two main cases: the entire set of real numbers $A = \mathbb{R} = (-\infty, \infty)$ and the unit interval $A = [0, 1]$. A prominent subcase, analyzed in Subsection 3.7 below, is that $x \in A = [0, 1]$ represents a mixture $xs_1 + (1-x)s_2$ of two pure strategies.

Time is also continuous, denoted by $t \in [0, \infty)$. At any particular time, the distribution of action choices within the population is represented by a cumulative distribution function $F(t, \cdot)$, where $F(t, x)$ denotes the fraction of the population choosing actions $y \leq x \in A$. The distribution $F(t, \cdot)$ encapsulates the present state of the system.

We note that a generic function $\psi(x, F(t, \cdot))$ encompasses quite a wide range of possibilities. It includes functions that only depend on $F(t, x)$. Also it includes functions that depend on the partial derivative¹ $F_x(t, x)$, since knowledge of the function $F(t, \cdot)$ specifies the partial derivative of F with respect to x (assuming it exists). So we not only include a large group of functions with local dependence (for example, those that depend on $F(t, x)$, $F_x(t, x)$, $F_{xx}(t, x)$, etc.), but, of course, we also include functions with non-local dependence, such as dependence on $F(t, x + 6.5)$ or on $F(t, y)$ over an interval of y values.

2.1. Payoff functions and landscapes

The population games we study in this paper arise from symmetric two-player games g in normal form, where $g(x, y) \in \mathbb{R}$ is the payoff to a player choosing action $x \in A$ when the other

¹ Throughout this paper, we will use subscript letters to denote partial derivatives. For example, $F_x(t, x) = \frac{\partial}{\partial x} F(t, x)$.

player chooses action $y \in A$. Classical game theory routinely extends the payoff function to mixtures in y chosen by the other player; one simply averages g over the mixture distribution. Similarly, it is natural to write the payoff ϕ of the population game obtained from g as the average, or “expected value”, of g given the current population distribution $F(t, \cdot)$:

$$\phi(x, F(t, \cdot)) = \int_A g(x, y) dF(t, y). \tag{1}$$

Here the Stieltjes integral, $\int_A g(x, y) dF(t, y)$, can, of course, be rewritten as $\int_A g(x, y) f(t, y) dy$ if the density $f(t, x) = F_x(t, x)$ exists.

The population dynamics presented in this paper hinge on V , the adjustment velocity function, which is the gradient of the payoff function ϕ ,

$$V(x, F(t, \cdot)) = \phi_x(x, F(t, \cdot)) = \int_A g_x(x, y) dF(t, y), \tag{2}$$

at any interior point x in A . For example, suppose that, at the current distribution of competitors’ prices, a firm finds that profit ϕ is locally an increasing function of its own price x , i.e., that $V > 0$. Then the firm will increase price, and do so more rapidly the steeper the profit gradient.

To ensure that V is well defined, we impose the restriction that, for any fixed y , the function g is absolutely continuous and piecewise smooth in x . This guarantees that g_x is defined almost everywhere, and that its righthand limit $\lim_{h \searrow 0} g_x(x + h, y)$ exists at all interior points x in A (and at the left endpoint when it is finite). To conform to the right continuity convention for cumulative distribution functions, we define $g_x(x, y)$ to be equal to this righthand limit on the set of measure zero where it is otherwise undefined. When the right endpoint of A is finite, its righthand limit does not exist, so there we define g_x to be the lefthand limit.

We also use (2) to define V at the left and right endpoints of A , except when this definition would cause population mass to escape from A . In that case we redefine V to equal 0. Specifically, if $\int_A g_x(x, y) dF(t, y) < 0$ at the left endpoint of A then we redefine V to equal 0 there, and if $\int_A g_x(x, y) dF(t, y) > 0$ at the right endpoint then we redefine V to equal 0 there.

Geometric intuition comes from thinking about the “landscape” at time t , which is the graph of ϕ at time t as a function of its first argument, x . Fig. 1 shows simple examples arising from $A = [0, 1]$ and $g(x, y) = (x - y)^2$. Suppose that the distribution at $t = 0$ is uniform over the interval $[0, 0.2]$. Then the landscape is $\phi(x, F(0, \cdot)) = \int_0^{0.2} (x - y)^2 5 dy = x^2 - 0.2x + 0.01\bar{3}$. Any action choice above $x = 0.1$ will be improved by increasing it, and the gradient $V = \phi_x$ steepens at larger choices of x . Hence players in this region have the incentive to increase x .

Now suppose that, at some much later time $t = 5$, we have half the players bunched at the right endpoint $x = 1$ and the other half at the left endpoint $x = 0$. The landscape now is $\phi(x, F(5, \cdot)) = \int_0^1 (x - y)^2 dF(5, y) = 0.5x^2 + 0.5(x - 1)^2$. Incentives are now symmetric to move away from the current mean $\mu^F = 0.5$, but at the current distribution, $F(5, \cdot)$, there is no room to do so. Hence that distribution should persist as a (stable) steady state.

Subsection 2.3 below will elaborate on this example, but for now we point out that the geometric intuition is quite general. When the current distribution $F(t, \cdot)$ is concentrated at a single point y_0 then the landscape is simply the relevant slice $\phi(x, F(t, \cdot)) = g(x, y_0)$ of the two-player payoff function. In general, the landscape is the weighted average of such slices, using $F(t, \cdot)$ as the weighting function over y . Varying $F(t, \cdot)$, even for a fixed g , clearly can generate a huge variety of landscapes.

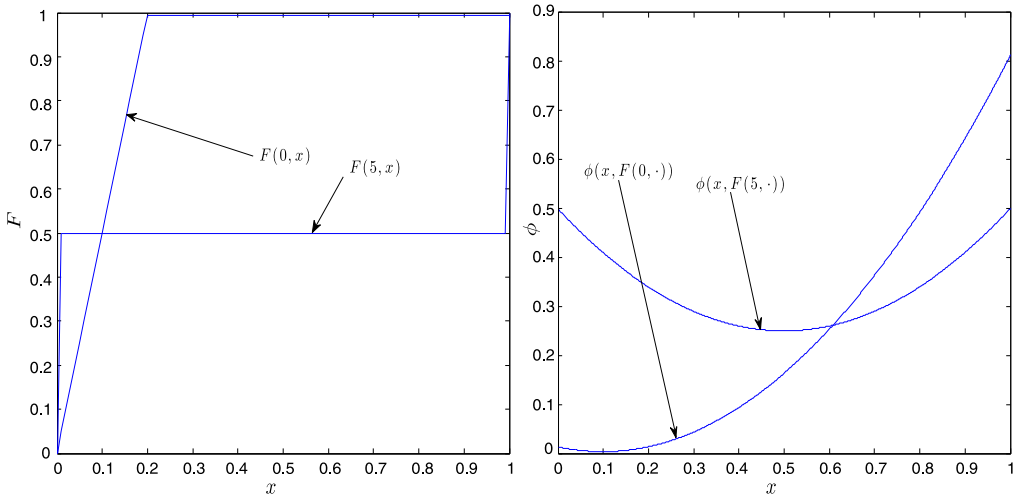


Fig. 1. Panel A. The cumulative distributions $F(0, x)$ and $F(5, x)$ described in the text. Panel B. The corresponding landscapes, $\phi(x, F(0, \cdot))$ and $\phi(x, F(5, \cdot))$.

The landscape’s slope, i.e., the payoff gradient V , is key, because players have the incentive to adjust their actions in order to move uphill. Of course, such adjustments change the distribution F , which alters the landscape. These alterations in turn will provoke further adjustments by the players and further alterations of the landscape. This dynamic process may or may not converge to a steady state.

2.2. The gradient dynamics PDE

To formalize this geometric intuition, we now derive equations characterizing how the current state $F(t, \cdot)$ changes over time. Since F depends on x as well as t , we will end up with partial differential equations, or in light of Eq. (2), integro-PDE’s.

The key behavioral assumption is that all players continuously adjust their actions so as to increase their own payoffs. More specifically, defining the payoff gradient V in (2) and then imposing the boundary constraints, we assume that each individual player systematically adjusts her choice $x \in A$ according to V evaluated at x . Specifically, we have the dynamics

$$dx = V dt. \tag{3}$$

Such behavior is sensible. Indeed, Theorem 1 of Friedman and Ostrov [13] demonstrates that, in the presence of quadratic adjustment costs, (3) describes behavior that is rational in an appropriate sense. It is also worth emphasizing that the information requirements are quite modest. Players must know their current action, x , and have a first-order sense of how their current payoff $\phi(x, F(t, \cdot))$ would change if x were changed slightly. Such an estimate of $V = \phi_x$ presumably comes from direct experience and does not require any knowledge of the underlying g or F , much less a conscious computation of the global payoff function $\phi(x, F(t, \cdot))$. In pricing a product, for example, the information requirement is that each firm knows its own current price and profit, and has some idea what its current profit would be if it chose a slightly different price. Firms don’t need to know the exact way that rivals’ prices affect their own demand, nor the current or likely future distribution of rivals’ prices, nor even the identity of rivals.

By conservation of population mass, our dynamics (3) imply the following fundamental PDE for the evolution of the density $f(t, x)$:

$$f_t = -(Vf)_x. \quad (4)$$

This is also a consequence of the Fokker–Planck–Kolmogorov equation, discussed briefly in Subsection 4.3 below.

Integrating (4) from $-\infty$ to x we obtain an alternative form of the PDE:

$$F_t = -VF_x. \quad (5)$$

To obtain intuition about this form, note that the left hand side of (5) evaluated at a point $x \in A$ is the rate of change in the fraction of the population that chooses actions $y \leq x$. The right hand side represents the flux, which is the adjustment speed times density of players moving their actions downward (hence the minus sign) from above x to below. Thus the equation says that the distribution changes via continuous adjustment by individual players—no population mass is gained or lost overall, and nobody jumps.

We will refer to either (4) or (5) as the *gradient dynamics PDE*.² Note the implicit assumption that the population is large. That all players face the same distribution $F(t, \cdot)$ presumes that each individual player has a negligible impact on the overall distribution.

Because the expression $V(x, F(t, \cdot))$ encompasses so many possibilities, statements about well-posedness, existence, and uniqueness are not possible for the general form of the gradient dynamics PDE. However, there are broad subclasses of V and subclasses of g (which define V through (2)), where we have well-posedness, existence, and uniqueness, given appropriate initial and boundary conditions. This paper explores a number of applications that fall into these subclasses. We will see that even when we restrict ourselves to subclasses where we know that a unique solution exists, those solutions can exhibit a remarkably wide variety of behavior.

2.3. A simple family of games with diverse behavior

Consider the parametrized family of pairwise payoff functions $g(x, y) = b|x - y|^a$ over the action space $A = [0, 1]$, where the parameter $a > 0$. When the parameter $b = 1$ (or any other positive number), the function g and the corresponding population payoff function ϕ embody congestion in the sense that each player has an incentive to choose x as distant as possible from other players' choices.

$\mathbf{g}(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2 = (\mathbf{x} - \mathbf{y})^2$: The consequences for $a > 1$ can be seen in the analytically tractable case $a = 2$ mentioned in Subsection 2.1 above. Let $\mu^F(t) = \int_0^1 y dF(t, y)$ and $(\sigma^F(t))^2 = \int_0^1 (y - \mu^F(t))^2 dF(t, y)$ denote the mean and variance of the distribution $F(t, \cdot)$. Then

$$\begin{aligned} \phi(x, F(t, \cdot)) &= \int_0^1 (x - y)^2 dF(t, y) = x^2 - 2x\mu^F(t) + [(\sigma^F(t))^2 + (\mu^F(t))^2] \\ &= (x - \mu^F(t))^2 + (\sigma^F(t))^2. \end{aligned} \quad (6)$$

² In physics and applied mathematics, these conservation of mass equations are referred to as *continuity equations*. When V has only local dependence on t and x , equations of the form (4) are called *conservation laws*, while equations of the form (5) are called *Hamilton–Jacobi equations*.

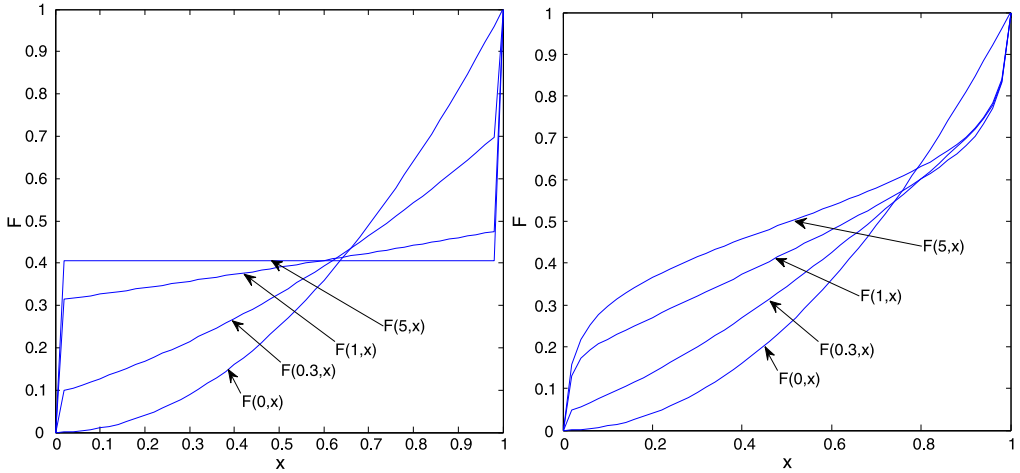


Fig. 2. Evolving population distributions for gradient dynamics on $A = [0, 1]$ with initial distribution $F(0, x) = x^2$. Panel A uses $g(x, y) = (x - y)^2$. By time $t = 5$, the state closely approximates a degenerate steady state distribution of the form $F = p\Theta_0 + (1 - p)\Theta_1$. Panel B uses $g(x, y) = |x - y|^{0.5}$. By time $t = 5$, the distribution closely approximates a non-degenerate steady state with higher density near the endpoints. See Appendix B for details on the numerical scheme used to compute these figures.

Clearly the population average payoff for (6) is maximized when the population mass is as far away as possible from the mean, and when the variance is as large as possible. This occurs when half the population mass is concentrated on each endpoint, as in the second distribution graphed in panel A of Fig. 1. Defining the Heaviside step function

$$\Theta_z(x) = \begin{cases} 0 & \text{if } x < z \\ 1 & \text{if } x \geq z, \end{cases}$$

the state that maximizes average payoff can be written $F^* = \frac{1}{2}\Theta_0 + \frac{1}{2}\Theta_1$.

However, panel A of Fig. 2 shows that other long run outcomes are possible. To analyze, first compute the gradient $V(x, F(t, \cdot)) = \phi_x(x, F(t, \cdot)) = 2(x - \mu^F(t))$ of (6), so the gradient dynamics PDE (5) reads

$$F_t(t, x) = 2(\mu^F(t) - x)F_x(t, x). \tag{7}$$

For $x \in (0, \mu^F(t))$ we have $F_t(t, x) > 0$ at points where the density, $f = F_x$, is positive, i.e., the accumulated population in $[0, x]$ increases, meaning mass moves to smaller x values. Eq. (7) similarly implies for each $x \in (\mu^F(t), 1)$ where the mass density is positive, the accumulated mass in $[x, 1]$ increases, meaning mass moves to larger x values. Thus, as illustrated in panel A of Fig. 2, all mass moves away from the current mean and towards the endpoints.

It follows that the distribution $F = p\Theta_0 + (1 - p)\Theta_1$ is a steady state for any $0 < p < 1$. These states are locally stable in the sense that any deviation that doesn't push mass past the mean $\mu^F(t) = p$ will shrink over time according to (7). Also, for any $z \in A$, the degenerate distribution Θ_z is a steady state, since $F_x(t, x) = 0$ for $x \neq z$ and $(\mu^F(t) - x) = 0$ for $x = z$, so in (7) we have $F_t(t, x) = 0$ everywhere. But these steady states are unstable, because small deviations to a non-degenerate distribution will, according to (7), increase over time.

$g(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$: The borderline case $a = 1$ is also analytically tractable. We split $\phi(x, F(t, \cdot)) = \int_0^1 |x - y| dF(t, y)$ into two integrals corresponding to the domain where y is

less than x and the domain where y is greater than x . We then apply integration by parts to both integrals, and differentiate with respect to x . This yields $V(x, F(t, \cdot)) = \phi_x(x, F(t, \cdot)) = 2F(t, x) - 1$. The median of F is, by definition, the solution to $2F(t, x) - 1 = 0$, so the gradient dynamics PDE simply says that mass flows away from the median. Thus, starting from any initial distribution with finite density at the median, the distribution converges to the maximal payoff state $F^* = \frac{1}{2}\Theta_0 + \frac{1}{2}\Theta_1$.

$g(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^{0.5}$: Numerical methods that will be explained in Section Appendix B are necessary to approximate the solution for non-integer values of a . As illustrated in panel B of Fig. 2, the steady state distributions are non-degenerate when $a < 1$. Why do we get dispersed steady states here but clumped steady states when $a > 1$? The key consideration is the *marginal* benefit of increasing distance from other players' chosen locations. For $a < 1$ that marginal benefit is greater for nearby locations so, loosely speaking, players seek to reduce the local density. For $a > 1$ that marginal benefit is greater for more distant locations, and distributions clumped at the boundary points allow the maximal distance from the distant players.

$g(\mathbf{x}, \mathbf{y}) = -|\mathbf{x} - \mathbf{y}|$: Now consider the case where $b = -1$, so $g(x, y) = -|x - y|^a$. Here, as with any negative b value, players want to be nearer to other players, not farther away, so we should expect that population mass will converge to an interior point in A . Yet behavior turns out to have some surprising aspects that depend on the value of the exponent a .

For the case $a = 1$, we have $g_x(x, y) = 1$ if $x < y$ and otherwise $g_x(x, y) = -1$, so here (2) gives us

$$\begin{aligned} V(x, F(t, \cdot)) &= \int_{-\infty}^x (-1) dF(t, y) + \int_x^{\infty} (1) dF(t, y) \\ &= -F(t, x) + (1 - F(t, x)) = 1 - 2F(t, x), \end{aligned}$$

and the gradient dynamics PDE (5) is

$$F_t + (1 - 2F)F_x = 0. \tag{8}$$

If the initial distribution is uniform on A , so $F(0, x) = x$ for all $x \in [0, 1]$, then (as can be verified by direct calculation of the partial derivatives) the solution to the given initial value problem for (8) when $t < \frac{1}{2}$ is

$$F(t, x) = \begin{cases} 0 & \text{if } x < t \\ \frac{x-t}{1-2t} & \text{if } t \leq x \leq 1-t \\ 1 & \text{if } x > 1-t. \end{cases} \tag{9}$$

In other words, the distribution remains uniform, but the width of its support shrinks linearly with time until, in the limit $t \nearrow \frac{1}{2}$, the distribution converges to the Heaviside step function $\Theta_{\frac{1}{2}}(x)$.

That is, the entire population mass converges to action $x = \frac{1}{2}$ at $t = \frac{1}{2}$.

What then? The nature of the game suggests that all players continue to choose $x = \frac{1}{2}$ when $t \geq \frac{1}{2}$, and this is borne out by the numerical simulation in panel A of Fig. 3, but there is a serious conceptual problem. When $t < \frac{1}{2}$, the solution is *classical*, i.e., the partial derivatives are defined and satisfy the PDE. However, when $t \geq \frac{1}{2}$, the distribution $F(t, x) = \Theta_{\frac{1}{2}}(x)$ is not continuous, much less differentiable, at $x = \frac{1}{2}$. Thus there is no classical solution after $t = \frac{1}{2}$, and it is not yet clear in what sense equation (8) is satisfied.

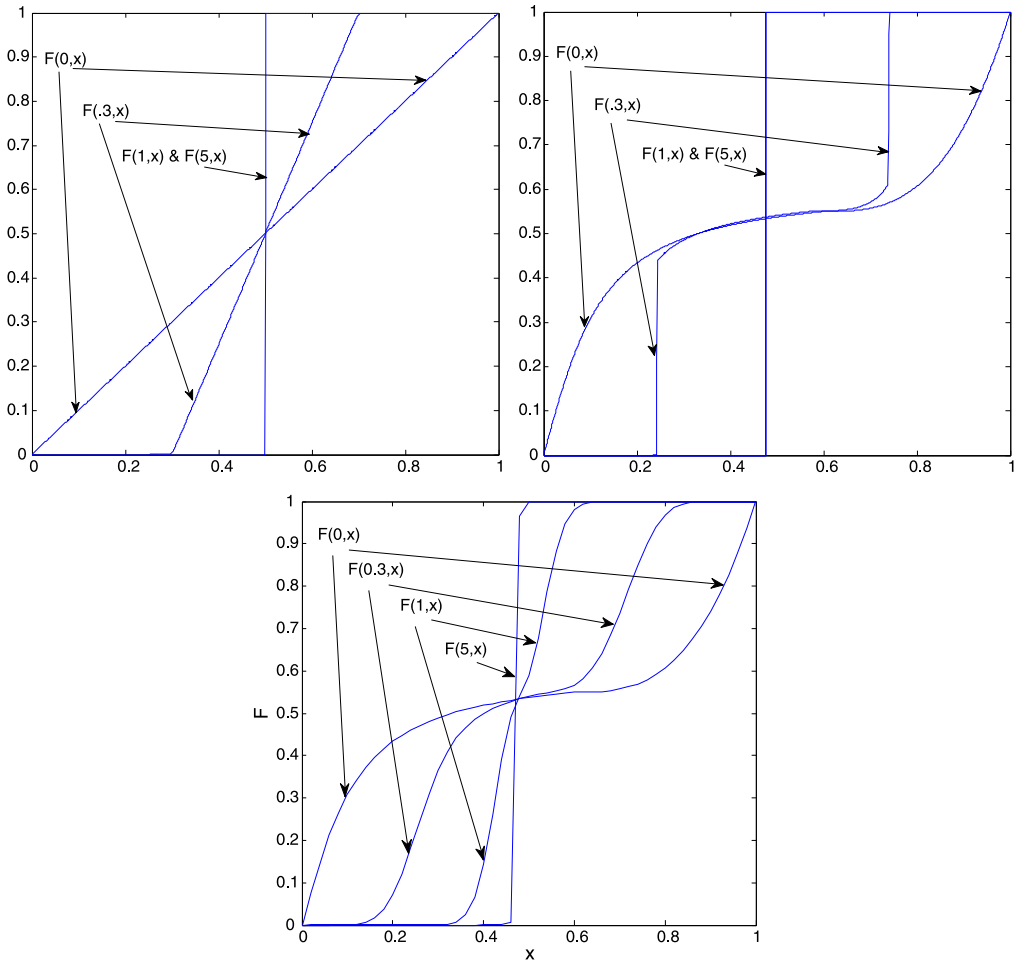


Fig. 3. Evolving population distributions for gradient dynamics on $A = [0, 1]$. Panel A uses $g(x, y) = -|x - y|$ and a uniform initial condition. The distribution reaches an exact steady state at $t = 0.5$, with all mass at $x = 0.5$. The discontinuity of F at $x = 0.5$ is called a *shock*. Panel B uses the same g but a different initial condition, $F(0, x)$. By $t = 0.3$, two shocks have appeared, and they travel until they merge at the initial median, near $x = 0.47$, before $t = 1$. Panel C uses $g(x, y) = -(x - y)^2$. Mass converges to a single point, but only asymptotically as $t \rightarrow \infty$, so a shock is not formed. See Appendix B for details on the numerical schemes used to compute these figures.

This evolution of discontinuities in the solution is not due to a special initial distribution. In fact, as illustrated in panel B of Fig. 3, a more typical smooth initial distribution leads to multiple discontinuities, and, further, these discontinuities travel over time. Nor is the problem due to a peculiar payoff function. Section 4 will show that discontinuities, called *shocks*, arise for a wide class of payoff functions. That section will show how to interpret the PDE and to find solutions for a wide class of functions where shocks arise and move over time.

$g(x, y) = -|x - y|^2 = -(x - y)^2$: One might guess that setting $b = -1$ is responsible for the occurrence of shocks, but that is not the case. When $g(x, y) = -|x - y|^2$, for a wide class of initial conditions, we are guaranteed to have smooth solutions at any later given time. These

smooth solutions can only converge to a degenerate steady state as $t \rightarrow \infty$ as shown in panel C of Fig. 3.

So the following questions are unavoidable when we explore gradient dynamics: How do we know when shocks can or cannot arise? How do we define the solution when they do arise? How do we find the solution analytically or simulate it numerically? We answer these questions in a number of contexts in the remainder of this paper.

3. Classical solutions and linear PDEs

For what classes of two-player games, $g(x, y)$, can we guarantee that shocks, that is, discontinuities in the solution, do not occur? In this section we show that when $A = \mathbb{R}$ it suffices for g to be smooth. We then explore the properties of some important polynomial g functions within this class for both $A = \mathbb{R}$ and $A = [0, 1]$.

3.1. Smooth g implies no shocks

In general, shocks can easily occur when the function V in (4) depends on $f(t, x)$, even when the dependence is quite smooth. But the global dependence of V on the distribution and a two-player game g via (2) provides special structure that often prevents shock formation. The following theorem shows that when the action space $A = \mathbb{R}$, shocks are impossible if g is sufficiently smooth:

Theorem 1. *Let $A = \mathbb{R}$ and let $g : A \times A \rightarrow \mathbb{R}$ have bounded second and third partial derivatives in its first argument. Also, let the initial density $f^0(x)$ have a bounded first derivative. Then neither the solution, $f(t, x)$, nor the partial derivative of the solution, $f_x(t, x)$, to the integro-partial differential equation defined by (2) and (4) can become unbounded in finite time.*

The proof appears in Subsection A.1 of Appendix A. It uses the bounds on g_{xx} and g_{xxx} to obtain bounds on f and f_x . This implies shocks cannot exist. That is, f cannot become unbounded nor discontinuous, or, equivalently from the cumulative distribution perspective, neither F nor F_x can become discontinuous. The proof uses the method of characteristics, which we describe in the next subsection.

Note in particular that the two cases $g(x, y) = \pm(x - y)^2$ discussed above in Subsection 2.3 fit the requirements of this theorem. The only remaining obstacle to applying Theorem 1 is the fact that $A = \mathbb{R}$ in the theorem, but $A = [0, 1]$ in Subsection 2.3. At the end of the next subsection, we will show how the method of characteristics also can be used to overcome this obstacle.

3.2. The method of characteristics

A standard technique for solving first order partial differential equations involves the equation's *characteristic curves* (also called *characteristics*). From each spatial location $x^0 \in A$ at $t = 0$, there is a characteristic curve $\xi(t)$ that evolves forward in time into the (t, x) plane. The solution can be determined along each of these characteristic curves, which can then be combined to form the full solution.

For our gradient dynamics PDE, let's first consider the case where the velocity field does not depend explicitly on the distribution. Specifically, we suppose there is a known, explicit function $v(t, x)$ which is twice differentiable in x , where $V(x, F(t, \cdot)) = v(t, x)$ for all t and x . The

gradient dynamics PDE (4) then becomes $f_t = -(v(t, x) f)_x$, which, upon applying the product rule to the right hand side and rearranging slightly, becomes

$$f_t + v f_x = -v_x f. \tag{10}$$

The evolution of the characteristic path and the solution along the characteristic path are given by the following system of ODEs along with their corresponding initial conditions:

$$\frac{d\xi(t)}{dt} = v(t, \xi(t)) \quad \xi(0) = x^0 \tag{11}$$

$$\frac{df(t, \xi(t))}{dt} = -v_x(t, \xi(t)) f(t, \xi(t)) \quad f(0, \xi(0)) = f^0(x^0). \tag{12}$$

The reason for this ODE system is made clear by (10): by choosing $\xi(t)$ so that it obeys $\frac{d\xi(t)}{dt} = v$ as given in (11), we see that the chain rule and (10) dictate the equation for $\frac{df(t, \xi(t))}{dt}$ given in (12). (Curves that do not follow (11) are not characteristic curves. Nor are they helpful, since they lead to ODE systems that are not closed and solvable.)

Note that the solution, f , at any specific point (\hat{t}, \hat{x}) depends upon the initial condition at only one point: the point at $t = 0$ to which (\hat{t}, \hat{x}) is connected by a characteristic curve. Put another way, information about the evolution of the solution of the gradient dynamics PDE travels out from the initial condition solely along the characteristics.

The term $v(t, x) f$ has a linear dependence on the unknown function f , which leads to the gradient dynamics PDE (10) being called a *linear* PDE. The ODEs, (11) and (12), are decoupled, so we first solve the ODE for $\xi(t)$ in (11) and use it to solve the ODE for $f(t, \xi(t))$ in (12), thereby obtaining the population density f along each characteristic:

$$f(t, \xi(t)) = f^0(x^0) e^{-\int_0^t v_x(s, \xi(s)) ds}. \tag{13}$$

In the case $A = \mathbb{R}$, Eq. (13) fully specifies the solution to the gradient dynamics PDE with given initial condition f^0 .

To illustrate, consider a payoff of the form $\phi = 1 - \frac{1}{2}(x - t + 1)^2$, a game against Nature in which a player’s payoff is higher the closer her action is to $x = t - 1$. Thus at time $t = 0$, the landscape is a parabolic hill with summit at $x = -1$, and over time the summit action shifts upward, e.g. to $x = 0$ at $t = 1$ and to $x = 1$ at $t = 2$. The gradient is $\phi_x = V = v(t, x) = t - x - 1$.

From (11) we have that $\frac{d\xi(t)}{dt} = t - \xi - 1$ and $\xi(0) = x^0$, so if $A = \mathbb{R}$, then

$$\xi(t) = t - 2 + (x^0 + 2)e^{-t}. \tag{14}$$

Since $v_x = -1$, Eq. (13) gives the solution along a characteristic as $f(t, \xi(t)) = f^0(x^0)e^t$, or, solving (14) for x^0 in terms of $x = \xi(t)$, the explicit solution to the gradient dynamics PDE is $f(t, x) = e^t f^0((x - t + 2)e^t - 2)$.

When the action space is $A = [0, 1]$, what happens when characteristics exit or enter at $x = 0$ and $x = 1$? In the current example, the slope $v(t, 0)$ of the characteristics at $x = 0$ is negative when $t < 1$ but positive when $t > 1$. Therefore, before $t = 1$, the characteristics intersect the $x = 0$ boundary. After $t = 1$, the characteristics switch direction and now move back into the $x \in (0, 1)$ region. What happens to the density f in either case is unclear for the moment.

To resolve the question, we use the alternative form of the gradient dynamics PDE given in (5),

$$F_t = -v F_x, \tag{15}$$

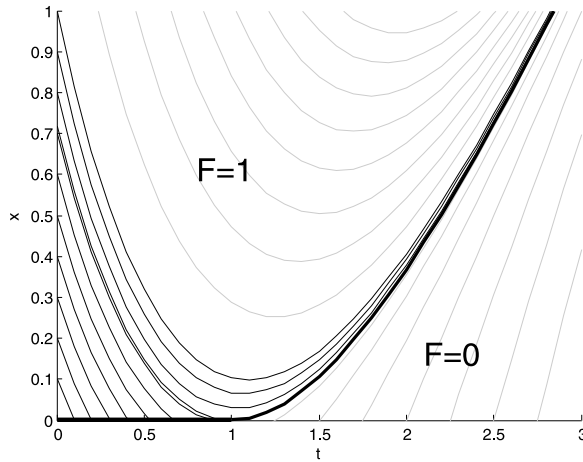


Fig. 4. Characteristics for gradient dynamics on $A = [0, 1]$ for the payoff function $\phi(t, x, f(\cdot)) = 1 - \frac{1}{2}(x - t + 1)^2$. On each gray characteristic curve at the bottom, we have $F = 0$. On each gray characteristic curve at the top, we have $F = 1$. On the black characteristic curves, the value of F is constant (and dictated by the initial condition). The bold curve accumulates mass as black characteristics collide into it until $t = 1$. After $t = 1$, the bold curve is a discontinuity in F , below which $F = 0$.

while imposing the natural boundary conditions³ $F(t, 0) = 0$ and $F(t, 1) = 1$. This yields the following ODEs for the characteristics:

$$\frac{d\xi(t)}{dt} = v(t, \xi(t)) \quad \xi(0) = x^0 \tag{16}$$

$$\frac{dF(t, \xi(t))}{dt} = 0 \quad F(0, \xi(0)) = F^0(x^0), \tag{17}$$

where the initial condition is $F^0(x) = \int_{-\infty}^x f^0(y) dy = \int_0^x f^0(y) dy$, since $A = [0, 1]$. Note from (17) that F stays constant along each characteristic, and therefore the full set of characteristics, as shown in Fig. 4, yields the solution for any given initial condition, $F(0, x)$. The interpretation from (17), but not clear from (11), is that the characteristics follow individual players' actions and players don't change their relative positions. For example, by following the characteristic $\xi(t)$ emanating from the x^0 where $F^0(x^0) = 0.75$, we follow the actions of the player at the 75th percentile. When V is a form that is not $v(t, x)$, as will be the case in Section 4, we do not always have that F is constant along characteristics, and so the interpretation of characteristics as action paths of individual players breaks down in the general case.

Now we return to our two cases, $g(x, y) = \pm(x - y)^2$, from Section 2.3. When $g(x, y) = (x - y)^2$, we have that $V = 2(x - \mu^F(t))$. Since $\mu^F(t) \in A = [0, 1]$, we cannot have $V > 0$ at $x = 0$, nor $V < 0$ at $x = 1$. That is, information from the boundaries, $x = 0$ and $x = 1$, cannot

³ The standard convention is that $F(t, x)$ is defined to be continuous from the right in x and have limits from the left in x . We generally follow this convention. However, when $A = [0, 1]$, we slightly abuse this convention by writing that $F(t, 0) = 0$, even if there is a finite mass at $x = 0$, which would mean that $\lim_{x \rightarrow 0^+} F(t, x) \neq 0$. Using $F(t, 0) = 0$ reflects the fact that in the PDE formulation, the convention is irrelevant and so the boundary condition $F(t, 0) = 0$ is correct, just as the boundary condition $F(t, 1) = 1$ is correct. Once the PDE is solved, we can again conform to the convention if desired by just redefining $F(t, x)$ to equal its limit from the right in x at all points of discontinuity, including at $x = 0$ if a finite mass is there.

have any effect on the solution on $(0, 1)$, the interior of A , at any time. The choice of $A = [0, 1]$, as opposed to $A = \mathbb{R}$, is irrelevant to the solution on $(0, 1)$, the interior of A , since it depends solely on the initial condition specified in $(0, 1)$. Therefore, we can apply **Theorem 1** and conclude that the solution cannot have shocks in the interior of A . Note that this conclusion only applies to the interior; mass can, and does, build on the boundaries of $A = [0, 1]$, as we have already seen in Subsection 2.3.

The reasoning for $g(x, y) = -(x - y)^2$ is slightly different. Here $V = -2(x - \mu^F(t))$ and so we cannot have $V < 0$ at $x = 0$, nor $V > 0$ at $x = 1$. This means that information from the boundaries (that is, from $F(t, 0) = 0$ and $F(t, 1) = 1$) definitely could affect the solution on the interior of A . But if we extend our domain to $A = \mathbb{R}$ and extend the initial condition so that $F(0, x) = 0$ when $x < 0$ and $F(0, x) = 1$ when $x > 1$, then the solution for $x \in (0, 1)$ is unaffected. This is because $V \geq 0$ when $x \leq 0$ and F doesn't change along characteristics, so $F(t, 0) = 0$ is guaranteed. Similar logic guarantees that $F(t, 1) = 1$. Given this equivalent formulation of the problem, **Theorem 1** guarantees that our solution has no shocks as long as the initial condition, $f^0(x)$, when extended to \mathbb{R} , has a bounded first derivative as required by the theorem. Similarly, as long as no shocks form over some initial time interval $[0, \hat{t}]$, if $f_x(\hat{t}, x)$ is bounded, then **Theorem 1** still applies to establish that no shock can form at later times $t > \hat{t}$. This later case corresponds to situation seen in panel C of Fig. 3 in Subsection 2.3. Since $f(0.3, x)$ has a bounded derivative in x , we can conclude that the convergence of mass to a point cannot occur in finite time, as that would imply the presence of a shock.

3.3. Quadratic g and their corresponding V functions

When $A = \mathbb{R}$ the method of characteristics allows us to solve the gradient dynamics PDE explicitly for quadratic two-player games, i.e., when $g(x, y) = ax + \frac{b}{2}x^2 + cxy + h(y)$. In this case $V(x, F(t, \cdot)) = \int_{-\infty}^{\infty} g_x(x, y) dF(t, y) = a + bx + c\mu^F(t)$, and we have the following key result:

Theorem 2. *Let*

$$g(x, y) = ax + \frac{b}{2}x^2 + cxy + h(y) \tag{18}$$

or let the velocity field take the form

$$V(x, F(t, \cdot)) = a + bx + c\mu^F(t), \tag{19}$$

where $x \in A = \mathbb{R}$, $t \geq 0$, and a, b , and c are real constants. Then, given an initial probability density, $f^0(x)$, with compact support, the gradient dynamics PDE (4) has the following unique solution:

$$f(t, x) = e^{-bt} f^0\left(\mu^F(0) + \left(x + \frac{a}{b+c}\right)e^{-bt} - \left(\mu^F(0) + \frac{a}{b+c}\right)e^{ct}\right) \tag{20}$$

and the mean is

$$\mu^F(t) = \left(\mu^F(0) + \frac{a}{b+c}\right)e^{(b+c)t} - \frac{a}{b+c}. \tag{21}$$

This solution is a classical solution if f^0 is differentiable. That is, the partial derivatives in the gradient dynamics PDE (4) are defined and satisfy this PDE at every point (t, x) when $t > 0$.

The proof is spelled out in Subsection A.2 of Appendix A.⁴ One integrates the gradient dynamics PDE to obtain an ODE for $\mu^F(t)$. Inserting its solution into the expression for $V(x, F(t, \cdot))$ yields an explicit function, $v(t, x)$. The result then follows from Eqs. (11) and (13).

3.4. Keynes' beauty contest

As a quick illustration of the economic implications of Theorem 2, recall that the winner in the famous beauty contest of Keynes [18] is the player who guesses most closely the average guess of all players. We consider a popular generalization in which the objective of every player is to guess a multiple $c > 0$ of the average guess $\mu^F(t)$. We assume a quadratic penalty for guessing incorrectly, and obtain the payoff function $\phi(x, F(t, \cdot)) = k - \frac{1}{2}(c\mu^F(t) - x)^2$, where k is a constant that has no effect on our analysis.

The game is tricky because most people think of a fixed distribution of play and choose c times the mean. For example, if $c = 2/3$ and players contemplate uniform choices over $A = [0, 1]$, then the mean would be 0.50 and their choices would cluster around 0.33. But in this case, a more sophisticated or experienced player would choose 0.22. If a player believes that others are sophisticated enough to choose 0.22, then she should choose about 0.14, etc.

We now apply gradient dynamics to the given payoff function. The gradient is

$$V(x, F(t, \cdot)) = c\mu^F(t) - x, \quad (22)$$

a special case of (19) with $a = 0$ and $b = -1$. Thus, if $A = \mathbb{R}$, (20) yields

$$f(t, x) = e^t f^0(\mu^F(0) + xe^t - \mu^F(0)e^{ct}). \quad (23)$$

Inspection of (23) reveals that the support of f shrinks as t increases, so we have clumping as $t \rightarrow \infty$ and all mass converges exponentially to $\mu^F(0)e^{(c-1)t}$. If $c = 1$, then convergence is to $\mu^F(0)$. If $c < 1$, the guesses converge to $x = 0$. If $c > 1$, the guesses diverge to ∞ if $\mu^F(0) > 0$ (or to $-\infty$ if $\mu^F(0) < 0$). Such behavior is broadly consistent with laboratory results since Nagel [22], who ran experiments with 15–18 paid human subjects, and a payoff function similar to ϕ with $c = 2/3$. See Fig. 5 for an example of the evolution of this game's solution according to gradient dynamics.

3.5. Cournot duopoly

As another application, consider the first mathematical model in economics, the Cournot (1838) duopoly. There are two firms that simultaneously choose output quantities x and y and face a linear demand function with slope scaled to -1 and intercept scaled to 1. The two firms have zero fixed costs, and identical constant marginal cost $m \in [0, 1)$. Then price is $1 - x - y$, and the restriction $x, y \in A = [0, 1]$ is natural. Unit profit is price minus marginal cost and the payoff function is quantity times unit profit, so $g(x, y) = x(1 - x - y - m)$. Eq. (18) holds with $a = 1 - m$, $b = -2$ and $c = -1$, and, were $A = \mathbb{R}$, Eq. (20) from Theorem 2 would give

$$f(t, x) = e^{2t} f^0 \left(0.5 + \left(x - \frac{1-m}{3} \right) e^{2t} - \left(0.5 - \frac{1-m}{3} \right) e^{-t} \right). \quad (24)$$

⁴ Subsection A.3 sketches how to generalize Theorem 2 to any $g(x, y)$ that is the sum of terms of the form $c_{nm}x^n y^m$, where c_{nm} are constants, $n = 1$ or 2, and m is any nonnegative integer.

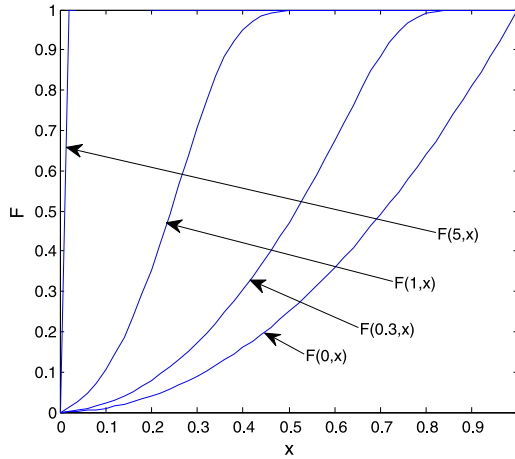


Fig. 5. Keynes beauty contest with $c = 2/3$. By time $t = 5$, the distribution closely approximates the unique steady state Θ_0 , i.e., all mass at $x = 0$. See Appendix B for details on this numerical computation.

But now consider a uniform initial distribution on $A = [0, 1]$ with $0 < m < \frac{1}{2}$. Since $V = (1 - m) - 2x - \mu^F(t)$ and, from (21), we have that $\mu^F(t)$ monotonically decreases from $\frac{1}{2}$ to $\frac{1-m}{3}$, we know that at all times $V > 0$ at $x = 0$ and $V < 0$ at $x = 1$. Therefore, by the argument in the last paragraph of Subsection 3.2, we can still apply (24), even though $A = [0, 1]$. In particular, as time t increases, the e^{2t} factors dominate, which implies the distribution remains uniform, but on an exponentially shrinking subinterval that always contains $x = \frac{1-m}{3}$.

There is nothing special about making the initial density uniform, however. For any reasonable initial distribution f^0 , the solution (24) converges exponentially to $\Theta_{\frac{1-m}{3}}$, the degenerate distribution with all mass concentrated at the Nash–Cournot equilibrium output quantity $x = \frac{1-m}{3}$. In some cases part of the population mass will build up at $x = 0$ — specifically, when the characteristics move from the interior of A to the boundary at $x = 0$ — before then moving towards $x = \frac{1-m}{3}$ — when the characteristics reverse direction and move from the boundary at $x = 0$ into the interior of A ; the underlying logic is exactly the same as in the example leading to Fig. 4 in Subsection 3.2.

We see three possible interpretations (none of them compelling) for the population game dynamics described by (24). Although g describes the rivalry between our focus firm and a single actual rival, the focus firm might face a large number of potential rivals whose output choices have distribution $F(t, \cdot)$. As firms marginally adjust output they see on average the local profit gradient ϕ_x , and therefore (24) describes the adjustment dynamics with the population of potential duopolists. A second interpretation is that the firm produces a wide variety of similar but distinct products and faces a different single rival for each product. It adjusts the output quantity for each product in response to experience gained on all products. Then $F(t, \cdot)$ describes the current cross-sectional distribution of rivals’ choices, which adjusts via (24). A third interpretation is that $\phi(x, F(t, \cdot))$ equals the expected value of $g(x, y)$ in the $dF(t, y)$ measure. This view represents the subjective expected profit of a firm contemplating (but not yet committed to) output x , where F summarizes management’s assessment of their rival’s possible future actions. Gradient dynamics represent an internal process of modifying beliefs as the firm contemplates the potential profit consequences to itself and its rivals, and only when the process converges to an invariant belief distribution does the firm actually commit to produce output.

3.6. An example from growth theory

At least since Romer [26], economists have modeled economic growth via complementarities and increasing returns technologies. Perhaps the simplest version is captured in the two person game $g(x, y) = xy - c(x)$, where the personal benefit xy is increasing in the other player’s choice $y \in A$ as well as one’s own choice $x \in A$, while the personal cost $c(\cdot)$ is assumed to be increasing and convex on $A = [0, \bar{m}]$, where $\bar{m} > 0$ represents the maximum feasible contribution to the public good.

It is natural to think of economic growth as a population game. Inserting g as above into Eqs. (1) and (2), we obtain the payoff function $\phi(x, F(t, \cdot)) = x\mu^F(t) - c(x)$ and its gradient

$$V(x, F(t, \cdot)) = \phi_x(x, F(t, \cdot)) = \mu^F(t) - c'(x). \tag{25}$$

Suppose first that marginal cost is constant, so $c'(x) = a_1 > 0$. If the initial mean $\mu^F(0) < a_1$, then thereafter $V \leq \mu^F(0) - a_1 < 0$ everywhere in A except at the lower endpoint $x = 0$, so the distribution converges in finite time to Θ_0 , where nobody contributes anything to the public good. On the other hand, if $\mu^F(0) > a_1$, then $V > 0$ everywhere on A except at the upper endpoint \bar{m} , so in finite time the distribution converges to $\Theta_{\bar{m}}$, where everyone contributes maximally. Thus constant marginal cost leads to extreme hysteresis, with the outcome depending entirely on whether the initial mean contribution exceeds or falls short of the threshold a_1 .

Behavior is quite different in the case of very convex cost, with $c'(0) = 0$ and $c'(x) \rightarrow \infty$ as $x \rightarrow \bar{m}$. Here the Intermediate Value Theorem and the convexity of c ensures that Eq. (25) has a unique interior root $z(t) \in A$. Clearly $V > 0$ in (25) if and only if $x < z(t)$, and $V < 0$ if and only if $x > z(t)$, so population mass moves towards $z(t)$. We should expect to see asymptotic convergence to a degenerate steady state distribution $\Theta_{\hat{z}}$, where \hat{z} is the unique root of $x - c'(x)$.

Such convergence can be shown explicitly in the case of quadratic cost, $c(x) = a_0 + a_1x + \frac{1}{2}a_2x^2$, when $A = \mathbb{R}$. Then (25) satisfies (19) with $a = -a_1, b = -a_2$ and $c = 1$, so (20) yields

$$f(t, x) = e^{a_2t} f^0 \left(\mu^F(0) + \left(x - \frac{a_1}{1 - a_2} \right) e^{a_2t} - \left(\mu^F(0) - \frac{a_1}{1 - a_2} \right) e^t \right), \tag{26}$$

and (21) yields

$$\mu^F(t) = \left(\mu^F(0) - \frac{a_1}{1 - a_2} \right) e^{(1-a_2)t} + \frac{a_1}{1 - a_2}. \tag{27}$$

Because $z(t) = \frac{\mu^F(t) - a_1}{a_2}$, Eq. (27) implies

$$z(t) = \frac{1}{a_2} \left(\mu^F(0) - \frac{a_1}{1 - a_2} \right) e^{(1-a_2)t} + \frac{a_1}{1 - a_2}.$$

Thus if $a_2 > 1$, we get enough convexity for the mass to converge exponentially to the mean, which exponentially converges to the finite value $\hat{z} = \frac{a_1}{1-a_2}$. On the other hand, if $a_2 < 1$ then (26)–(27) say that population mass diverges exponentially either to $+\infty$ or to $-\infty$, depending on the sign of $(\mu^F(0) - \frac{a_1}{1-a_2})$. For $A = [0, \bar{m}]$, now-familiar reasoning tells us that convergence is to the upper endpoint $x = \bar{m}$ or the lower endpoint $x = 0$ when $a_2 < 1$, and also when $a_2 > 1$ but $\hat{z} \notin A$.

3.7. The bilinear case

Consider now the special case that $x \in A = [0, 1]$ represents the mixture $xs_1 + (1 - x)s_2$ of two pure strategies, s_1 and s_2 . Let m_{ij} be the payoff to a player using s_i when matched with another player using s_j for $i, j = 1, 2$. We will say that g is the *mixed extension* of the symmetric two-player game $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ if it is the payoff given the player's mix x and the opponent's independent mix y , i.e., if

$$g(x, y) = \mathbf{x}^T M \mathbf{y} = xym_{11} + (1 - x)ym_{21} + x(1 - y)m_{12} + (1 - x)(1 - y)m_{22}, \tag{28}$$

where $\mathbf{x}^T = [x, 1 - x]$ and $\mathbf{y}^T = [y, 1 - y]$. Defining the composite parameters $m_1 = m_{11} - m_{21}$ and $m_2 = m_{22} - m_{12}$, we have

$$\begin{aligned} g_x(x, y) &= (1, -1)M\mathbf{y} = ym_{11} - ym_{21} + (1 - y)m_{12} - (1 - y)m_{22} \\ &= (m_1 + m_2)y - m_2, \end{aligned} \tag{29}$$

and so the corresponding population game has gradient

$$V(x, F(t, \cdot)) = \phi_x(x, F(t, \cdot)) = \int_0^1 g_x(x, y) dF(t, y) = (m_1 + m_2)\mu^F(t) - m_2. \tag{30}$$

Thus, in the interior of A , the slope V of each characteristic curve is independent of x and depends only on the mean, $\mu^F(t)$. That is, these bilinear payoff functions g produce landscapes with constant slope, and the state evolves via *censored translation* — that is, the distribution slides rigidly left or right with velocity V until it hits an endpoint, where mass piles up. By (30), that velocity is zero for any distribution $F(t, \cdot)$ such that $\mu^F(t) = \mu^*$, where $\mu^* = \frac{m_2}{m_1 + m_2}$, so such distributions are steady states. Moreover, inspection of (30) reveals that $\mu(t)$ will move towards μ^* if $(m_1 + m_2) < 0$, but will move away from $\mu(t)$ if $(m_1 + m_2) > 0$. Of course, by its definition, $\mu^* \in (0, 1)$ iff m_1 and m_2 have the same sign.

Once again (30) is a case of (19), here with $a = -m_2$, $b = 0$ and $c = m_1 + m_2$. Therefore, ignoring censoring, Eqs. (20)–(21) yield the exact solution

$$f(t, x) = f^0(x + (\mu^F(0) - \mu^*)(1 - e^{(m_1 + m_2)t})), \tag{31}$$

with mean

$$\mu^F(t) = (\mu^F(0) - \mu^*)e^{(m_1 + m_2)t} + \mu^*. \tag{32}$$

This conforms to our intuitions. If $(m_1 + m_2) < 0$, the solution just translates monotonically so that $\mu^F(t)$ converges exponentially to μ^* . If $(m_1 + m_2) > 0$, the solution translates off to ∞ if $\mu^F(0) > \mu^*$, or off to $-\infty$ if $\mu^F(0) < \mu^*$.

Taking censoring into account, i.e., mass building up at an endpoint, does not drastically alter the conclusions. Since $\mu^F(t)$ never crosses μ^* , Eq. (30) tells us that either $V \leq 0$ for all t and x or $V \geq 0$ for all t and x . It follows that mass can only build up at one of the two endpoints and any mass that builds up at an endpoint will always stay at that endpoint. See Fig. 6, for example.

Now we can characterize behavior quite precisely. When $\mu^* \in (0, 1)$ and $(m_1 + m_2) < 0$, the distribution converges exponentially via censored translation to a distribution with mean μ^* . When $(m_1 + m_2) > 0$, all mass reaches the right endpoint $x = 1$ in finite time if $\mu^F(0) > \mu^*$ (and a fortiori if μ^* is negative), and all mass reaches the left endpoint $x = 0$ in finite time if $\mu^F(0) < \mu^*$ (and a fortiori if $\mu^* > 1$).

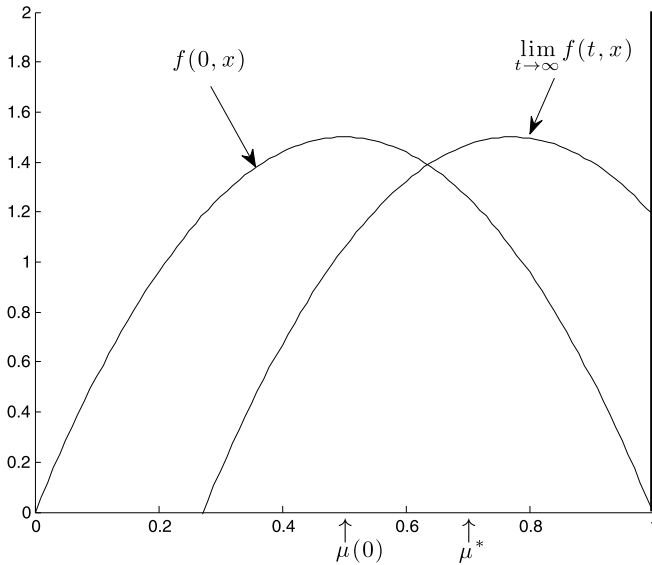


Fig. 6. Initial density and asymptotic distribution for bilinear example with $m_1 = -3, m_2 = -7$. Note that some of the mass accumulates at $x = 1$ in the asymptotic distribution.

Thus we have proved the following theorem:

Theorem 3. Let $\phi(x, F(t, \cdot)) = \int_0^1 g(x, y) dF(t, y)$ where g is the mixed extension of $M = ((m_{ij}))_{i,j=1,2}$, as defined in (28), and let F^0 be an arbitrary initial state. Then the solution $F(t, \cdot)$ to the gradient dynamics PDE (5) converges monotonically to an asymptotic distribution $F^* = \lim_{t \rightarrow \infty} F(t, \cdot)$ that depends on the composite parameters as follows:

- (a) If $m_1, m_2 < 0$ then F^* is the censored translation of F^0 with mean $\mu^* := \frac{m_2}{m_1+m_2}$, and the convergence of $F(t, \cdot)$ to F^* is exponential.
- (b) If $m_1, m_2 > 0$ then $F^* = \Theta_1$ (or $F^* = \Theta_0$) whenever $\mu_{F^0} >$ (or $\mu_{F^0} <$) $\mu^* = \frac{m_2}{m_1+m_2}$, and in either case convergence is complete in finite time.
- (c) If $m_1 > 0 > m_2$ (or $m_1 < 0 < m_2$) then $F(t, \cdot) \rightarrow F^* = \Theta_1$ (or $F(t, \cdot) \rightarrow F^* = \Theta_0$) in finite time.

Case (a) of the theorem occurs in 2×2 games like Hawk-Dove or Chicken. As illustrated in Fig. 6, the initial distribution shifts up (if $V > 0$ because the initial mean is below μ^*) or shifts down (if $V < 0$ because the initial mean is above μ^*) until $\mu^{F(t)}$ hits μ^* . The shift slows as the current mean approaches μ^* and V approaches zero.⁵

⁵ This suggests a resolution to a conundrum in evolutionary game theory. The standard prediction is that mean play converges to μ^* , but one modeling approach (called monomorphic) assumes that all players use the same mixed strategy μ^* , while another approach (called polymorphic) assumes that a fraction μ^* of players use the first pure strategy and the other players all use the second pure strategy, and nobody mixes. Our approach predicts that the initial dispersion of mixes will persist, but the mean will shift over time in the appropriate direction. The laboratory data seem roughly consistent with our prediction.

Case (b) occurs in 2×2 games like Coordination or Stag Hunt. Here any discrepancy between μ^* and the initial mean increases over time, and V increases as well. Hence we get convergence in finite time of all population mass to the pure strategy that initially had a higher expected payoff.

Case (c) occurs in 2×2 games with dominant strategies, e.g., Prisoner’s dilemma. Here the expression for μ^* lies outside $A = [0, 1]$, and in finite time all population mass piles up at the endpoint corresponding to the dominant strategy.

4. Solutions with shocks and nonlinear PDEs

Behavior is especially interesting with two-player payoffs $g(x, y)$ that yield nonlinear gradient dynamics PDEs, as their solutions are generally not smooth. (Note from Theorem 1 that since the solution is not smooth, the payoff function $g(x, y)$ cannot be smooth.) To describe population behavior arising from these games, we first derive equations for the characteristic curves — now generalized for a broad class of nonlinear PDE — and illustrate their implications using the Bertrand pricing model. Then, within a more specific context, we show what happens when characteristics collide and form shocks. The rest of the section applies this knowledge to a model of financial market dynamics.

4.1. Characteristics for nonlinear PDE

We consider two-player games $g(x, y)$ that yield nonlinear gradient dynamics PDEs of the form

$$F_t(t, x) + G(t, x, F(t, x), F_x(t, x)) = 0. \tag{33}$$

Note from (5) that $G = VF_x$ where V , in addition to depending on t and x as before, can now also depend on $F(t, x)$ and $F_x(t, x)$.⁶ Specification is completed by an initial distribution $F(0, x) = F^0(x)$ for all $x \in A$, together with the value of $F(t, x)$ at any boundary points x of A for all $t \in [0, \infty)$. To focus on the main new issues, we will assume for the moment that there are no boundary points, so $A = \mathbb{R}$.

To solve such an initial value problem, we must extend the method of characteristics from the linear case, given in (16) and (17), to our new nonlinear context. As before, we let $x = \xi(t)$ denote a characteristic path in the (t, x) plane emanating from some initial point $(0, x^0)$. Now let $u(t) = F(t, \xi(t))$ and $p(t) = F_x(t, \xi(t))$ denote the values of F and F_x along this characteristic. Using this notation, the PDE (33) tells us that $F_t(t, \xi(t)) + G(t, \xi(t), u(t), p(t)) = 0$ on this characteristic.

The system of ODEs defining any characteristic curve for (33) and the behavior along it is

$$\frac{d\xi}{dt} = G_{(F_x)}(t, \xi, u, p) \qquad \xi(0) = x^0 \tag{34}$$

$$\frac{du}{dt} = pG_{(F_x)}(t, \xi, u, p) - G(t, \xi, u, p) \qquad u(0) = F^0(x^0) \tag{35}$$

$$\frac{dp}{dt} = -G_x(t, \xi, u, p) - pG_F(t, \xi, u, p) \qquad p(0) = [F^0]_x(x^0), \tag{36}$$

⁶ This formulation is general enough to include all the applications we consider in the present paper. This is fortunate, as even extending the formulation to include dependence on F_{xx} can lead to difficulties. For example, when $G(F_{xx}(t, x)) = F_{xx}(t, x)$, the gradient dynamics equation (33) becomes the “backwards heat equation”, which is known to be ill-posed when it tries to evolve forward in time (e.g., Guenther and Lee [14]).

where $G_F(t, \xi, u, p)$ represents the partial derivative with respect to u , the third argument, and $G_{(F_x)}(t, \xi, u, p)$ represents the partial with respect to the fourth argument p . To see why this is the correct system of ODEs, differentiate the definition of $u(t)$ using the chain rule and then substitute using (33) and (34) to obtain (35). To obtain (36), first differentiate the definition of $p(t)$ using the chain rule, to get $\frac{dp}{dt} = F_{xt} + F_{xx} \frac{d\xi}{dt} = [F_t]_x + F_{xx} \frac{d\xi}{dt}$. Next, substitute $F_t = -G$ from (33), and apply the chain rule to the x differentiation of G to get $\frac{dp}{dt} = -G_x - G_F F_x - G_{(F_x)} F_{xx} + F_{xx} \frac{d\xi}{dt}$. Now (34) implies (36). This last step explains why (34) is the correct ODE for the characteristic path $\xi(t)$ — it allows us to cancel the F_{xx} terms, thus making (34)–(36) a closed system of ODEs in t, ξ, u , and p .

Finally, we verify that (34)–(36) generalizes the characteristic equations (16) and (17) from the linear PDE case. We have $G = v(t, x)F_x$ in the linear case, so Eq. (34) becomes $\frac{d\xi}{dt} = v(t, x)$, which is Eq. (16), while Eq. (35) becomes $\frac{du}{dt} = pv - vp = 0$, which is Eq. (17). Eq. (36) for $\frac{dp}{dt}$ is unnecessary in the linear case, as (16) and (17) form a closed system in t, ξ , and u , which is solvable without knowledge of p .

The method of characteristics in the nonlinear case (33) can be used to generate the solution forward in time as long as F is twice continuously differentiable. In the linear case when $A = \mathbb{R}$, we can guarantee that degree of smoothness by making $v(t, x)$ and the initial condition sufficiently smooth. By contrast, in the nonlinear case, regardless of how smooth G in (33) and the initial condition F^0 are, F or F_x can blow up in finite time as characteristic curves collide, causing F or F_x to become discontinuous. We give an example in the next subsection.

4.2. Preemption and Bertrand pricing

Consider the following foraging scenario, based on Rogers [25]. A resource (say, a berry patch) has gross value 1 when fully ripe and a lesser value x at earlier times, scaled so that $x \in [0, 1]$ represents time as well as resource value. By incurring a specified cost $c \in [0, 1)$, a player (or forager) can visit and attempt to harvest the resource at any value (or time) x of her choosing. In the two-player version of the game, a second player has exactly the same opportunity, but only the first to visit the resource gains the gross value x ; the latecomer receives nothing.

Thus we have the pairwise payoff function $g(x, y) = x - c$ if $x < y$ and $= 0$ if $x > y$; for completeness say that $g(x, y) = (x - c)/2$ if $x = y$. The corresponding population game has payoff $\phi(x, F(t, \cdot)) = \int_0^1 g(x, y) dF(t, y) = (x - c) \int_x^1 dF(t, y)$, with gradient

$$V(x, F(t, \cdot)) = \phi_x(x, F(t, \cdot)) = 1 - F(t, x) - (x - c)F_x(t, x), \tag{37}$$

and so the gradient dynamics PDE (5) becomes

$$F_t + F_x - FF_x - (x - c)(F_x)^2 = 0. \tag{38}$$

These dynamics can be thought of as taking place in evolutionary time, in which the forager species adapts its behavior in response to accrued fitness. Thus, if x is thought of as time rather than ripeness, it runs on a time scale incomparably faster than adaptation time t .⁷

Eq. (38) is of the form (33) with $G(t, \xi, u, p) = p - up - (\xi - c)p^2$, a nice, smooth polynomial. Characteristic equation (36) now is:

⁷ Another interpretation is Bertrand duopoly with inelastic demand for a single unit. Here x and y are prices, c is avoidable cost, and g is profit in the two-player game. The corresponding population game has the same interpretations as it does for the Cournot model.

$$\frac{dp}{dt} = 2(p(t))^2 \quad p(0) = [F^0]_x(x^0). \tag{39}$$

Assume that the initial distribution is smooth, take any starting point $x^0 \in A$, and let $p^0 = p(0) = [F^0]_x(x^0)$ be the density at this point. The solution to (39) is $p(t) = \frac{p^0}{1-2tp^0}$, which blows up at $t = \frac{1}{2p^0}$. That is, despite the smooth initial condition and G function, there is a singularity in the solution at $t = \frac{1}{2p^0}$, unless the chosen characteristic has already collided with another characteristic. If it has collided with another characteristic, the point in the (t, x) plane where the collision occurs will be part of a *shock curve*. A shock curve, often simply called a *shock*, is a curve of discontinuity in F or F_x .⁸ The next subsection further explores shocks within an important and useful class of PDEs called balance equations.

4.3. Balance equations

To understand examples of what happens at shocks, consider the following class of nonlinear PDE, which is somewhat less general than the form in (33):

$$F_t + [H(t, x, F(t, x))]_x = Q(t, x, F(t, x)). \tag{40}$$

PDEs of this form are called *balance equations* in the PDE literature, where H is called the *flux function* and Q is called the *source term*. The characteristic equations (34) and (35) for the balance equation (40) are

$$\frac{d\xi}{dt} = H_F(t, \xi, u) \quad \xi(0) = x^0 \tag{41}$$

$$\frac{du}{dt} = -H_x(t, \xi, u) + Q(t, \xi, u) \quad u(0) = F^0(x^0). \tag{42}$$

Note that, as in the linear case, the right hand sides of these two equations do not involve p . It again follows that we do not need Eq. (36) for $\frac{dp}{dt}$ to specify the characteristic ξ , or the solution u along the characteristic.

As with the general nonlinear PDE form in (33), characteristics for the balance equation (40) can easily collide, which form shocks. As colliding characteristics for balance equations carry different values of F , shocks represent discontinuities in the distribution, F .

To illustrate, recall from Subsection 2.3 the two-player game $g(x, y) = -|x - y|$, which yielded the gradient dynamics PDE $F_t + (1 - 2F)F_x = 0$. Rewritten in balance equation form, this is

$$F_t + [H(F)]_x = 0, \quad \text{where } H(F) = F - F^2. \tag{43}$$

The corresponding characteristic equations (41) and (42) are

$$\frac{d\xi}{dt} = 1 - 2u(t) \quad \xi(0) = x^0 \tag{44}$$

$$\frac{du}{dt} = 0 \quad u(0) = F^0(x^0). \tag{45}$$

⁸ Another interesting behavior can arise in this model when A has one or more boundary points \hat{x} . The slope of the characteristics emanating from the initial condition as $x^0 \rightarrow \hat{x}$ may not agree with the slope of the characteristics emanating from the boundary (t, \hat{x}) as $t \rightarrow 0$. This can give rise to a “rarefaction fan” as explained in, e.g., Friedman and Ostrov [13], Evans [11], and Smoller [32].

Eq. (45) tells us that $u(t) = F^0(x^0)$; that is, the value of F remains constant along each characteristic, and so (44) says that the characteristics are straight lines whose slopes are $1 - 2F^0(x^0)$.

Suppose that $A = [0, 1]$. Since $F = 0$ at $x = 0$, we have that all the characteristics that emanate from the boundary $x = 0$ (for all $t \in [0, \infty)$) have a constant slope of 1 and that $F = 0$ on them. Similarly, all the characteristics emanating from the boundary $x = 1$ have a constant slope of -1 and $F = 1$ on them. So, given the uniform initial distribution $F^0(x) = x$ assumed in Subsection 2.3, the characteristic equations (44) and (45) clearly generate the solution given in (9), which is valid until all the characteristics emanating from the initial condition collide at $t = 1/2$.

But what happens after $t = 1/2$? Here, the meaning of the PDE is not immediately clear, nor is the uniqueness of the solution once we allow for discontinuities. As shown in Friedman and Ostrov [13], we give a meaning to any balance equation PDE and retain a unique solution when $t \geq 1/2$ by understanding that the dynamics equation (3) used in Subsection 2.2 to derive our PDE assumes perfect information on the part of the players. But information is unlikely to be perfect, and this can be modeled by adding a stochastic noise component to (3):

$$dx = V dt + \sigma dB,$$

where dB is the Brownian motion differential and σ is a constant representing the degree of imperfect information. By understanding that the solution to our first order PDE must represent the solution in the limit as $\sigma \rightarrow 0$, we recover a unique solution, as is well known by researchers in PDE.

The literature shows (see Friedman and Ostrov [13], Dafermos [9], or Smoller [32], for example) that this limiting solution for balance equations is uniquely defined by two restrictions on the shocks:

1. The *entropy condition* must be satisfied at all shocks. That is, characteristic curves can terminate on shocks but cannot emanate from shocks. The idea is that information about the solution, which evolves via the characteristics, can be destroyed, but not created, at shocks.
2. The *Rankine–Hugoniot jump condition* must be satisfied at all shocks. That is, at any shock curve $x = s(t)$, the slope $\frac{ds}{dt}$ must satisfy

$$\frac{ds}{dt} = \lim_{\varepsilon \rightarrow 0^+} \frac{H(t, s(t), F(t, s(t) + \varepsilon)) - H(t, s(t), F(t, s(t) - \varepsilon))}{F(t, s(t) + \varepsilon) - F(t, s(t) - \varepsilon)}. \tag{46}$$

The Rankine–Hugoniot condition follows from the fact that F is a *weak* (or *generalized* or *distributional*) solution in the following sense: As explained in the literature cited above, we can multiply the balance equation by a smooth test function and then integrate over a neighborhood of the (t, x) plane that includes any shock. Then we can use integration by parts to transfer the partial derivatives to the test function, so that no derivatives of F remain. The solution is weak in that we require the resulting integral equation to be satisfied instead of the PDE itself. The Rankine–Hugoniot condition is a direct consequence of that integral equation.

Returning to the example, the unique solution requires that for $t \geq \frac{1}{2}$ there is a single shock where $F = 0$ below the shock and $F = 1$ above the shock so, by the Rankine–Hugoniot condition, $\frac{ds}{dt} = \frac{(1-1^2)-(0-0^2)}{1-0} = 0$. Thus the shock curve must be $s(t) = \frac{1}{2}$ for $t \geq \frac{1}{2}$. That is, in the unique solution, when $t \geq \frac{1}{2}$, we have that $F(x, t) = 0$ if $x < \frac{1}{2}$ and $F(x, t) = 1$ if $x \geq \frac{1}{2}$. Note that it is the characteristics which emanate from the two boundaries that terminate on this shock when $t > \frac{1}{2}$.

4.4. Financial Market Shocks

To demonstrate the application of these techniques, we develop and analyze a model of a financial market populated by fund managers. Each manager chooses leverage $x \in A$, where $x < 0$ indicates short selling of the risky asset (or market portfolio) and $x > 1$ indicates borrowing the safe asset in order to buy more of the risky asset. The probability that the manager goes bankrupt and the associated bankruptcy costs both are approximately linear in $|x|$, so investors using a manager with leverage x incur a quadratic cost approximated by $0.5cx^2$ for some fixed parameter $c > 0$.

The model treats as exogenous the returns $r(t)$ that the risky asset provides in excess of the safe return. A crude empirical proxy is the yield over the last 12 months on the S&P 500 index less the 10 year Treasury Bill annual yield; an exponential average of the daily yield differential is a more refined proxy. Historically, $r(t)$ has averaged about 5% per annum, but has sometimes been negative. As far as the model is concerned, the relevant point is that (on average, abstracting away from the manager’s “alpha”, i.e., idiosyncratic skill or luck) a manager choosing leverage x receives gross return $xr(t)$.

The game the managers play hinges on *relative* performance. Higher performance rank brings bonuses and competing job offers, and also increases managers’ compensation by attracting more investor funds. The importance of relative performance is widely recognized (e.g., the financial press prominently publishes quarterly rankings) and is documented empirically in articles such as [4,30,17]. However, its consequences are seldom modeled formally, as we shall now do.

Del Guercio and Tkac [10], among others, document the fact that relatively low performance is more damaging to managers than relatively high performance is beneficial. To capture this empirical asymmetry (which may ultimately be due to “loss aversion” or similar behavioral effects), we shall say that, relative to a rival manager with leverage y , a manager choosing x obtains “pride” component $g^P(x, y) = \max\{0, (x - y)r(t)\}$, and “envy” component $g^E(x, y) = \min\{0, (x - y)r(t)\}$. The overall two-player game payoff is $g = ag^E + bg^P$, with asymmetric weights $a > b \geq 0$.

Assume for the moment that there are no constraints on leverage, so the players’ action space is $A = \mathbb{R}$. When $r(t) > 0$, the population game payoff arising from g is

$$\begin{aligned} \int_A g(x, y) dF(t, y) &= r(t) \left[b \int_{-\infty}^x (x - y) dF(t, y) + a \int_x^{\infty} (x - y) dF(t, y) \right] \\ &= r(t) \left[a \int_{-\infty}^{\infty} (x - y) dF(t, y) + (b - a) \int_{-\infty}^x (x - y) dF(t, y) \right] \\ &= r(t) \left[a(x - \mu^F(t)) + (b - a) \int_{-\infty}^x F(t, y) dy \right]. \end{aligned}$$

The last expression uses integration by parts, assuming that the support of the current distribution has a finite lower bound. The expression is the same when $r(t) < 0$ except that a and b switch roles. Thus, including the risk cost $0.5cx^2$, the payoff function, ϕ , in the population game is

$$\begin{aligned} \phi(x, F(t, \cdot)) &= [b + (a - b)\Theta_0(r(t))](x - \mu^F(t))r(t) \\ &\quad - (a - b)|r(t)| \int_{-\infty}^x F(t, y) dy - 0.5cx^2 \end{aligned} \tag{47}$$

(note that $b + (a - b)\Theta_0(r(t))$ equals a when $r(t) > 0$ and equals b when $r(t) < 0$) and the gradient of ϕ is

$$\begin{aligned} V(x, F(t, \cdot)) &= \phi_x(x, F(t, \cdot)) \\ &= [b + (a - b)\Theta_0(r(t))]r(t) - (a - b)|r(t)|F(t, x) - cx. \end{aligned} \tag{48}$$

To complete the model, we assume gradient dynamics. In addition to general intuitive appeal, they can be justified quite specifically in the present setting. The well-known “price pressure” effect increases per-share trading cost linearly (to a good approximation over a substantial range) in the net amount traded in a given short time interval. It follows that the adjustment cost (net trade times per share trading cost) is quadratic. Theorem 1 of Friedman and Ostrov [13] therefore applies, and states that in an appropriate sense it is optimal for managers to adjust leverage at a rate proportional to the payoff gradient.

The gradient dynamics PDE (5) can be written in balance equation form (40)

$$F_t + [H(t, x, F)]_x = Q(F), \tag{49}$$

where

$$H(t, x, F) = [b + (a - b)\Theta_0(r(t))]r(t)F - \frac{1}{2}(a - b)|r(t)|F^2 - cxF \tag{50}$$

$$\text{and } Q(F) = -cF, \tag{51}$$

so the characteristic equations (41) and (42) are

$$\frac{d\xi}{dt} = [b + (a - b)\Theta_0(r(t))]r(t) - (a - b)|r(t)|u(t) - c\xi(t) \quad \xi(0) = x^0 \tag{52}$$

$$\frac{du}{dt} = 0 \quad u(0) = F^0(x^0). \tag{53}$$

Eq. (53) tells us that $u = F$ is constant along characteristics, so each characteristic follows the leverage path of a particular manager.

A simple analysis of Eq. (52) shows that shocks are inevitable when $r(t)$ remains bounded away from 0. Consider any two characteristics, $\xi_1(t)$ and $\xi_2(t)$, that start apart; say $\xi_1(0) < \xi_2(0)$. Because F is monotonically increasing, we have that $u_1 \leq u_2$. To see that characteristics collide, suppose that $u_1 < u_2$. Since $a - b > 0$ and $r(t)$ is bounded away from 0, (52) shows that as long as $\xi_1(t) \leq \xi_2(t)$, then $\frac{d(\xi_2 - \xi_1)}{dt} < -k < 0$ for some constant k , which guarantees a shock by time $t = \frac{\xi_2(0) - \xi_1(0)}{k}$. The interpretation is that a more leveraged manager increases her leverage more slowly (or decreases it more rapidly) than her less leveraged peers, until her characteristic collides with her peers’, and thereafter they follow exactly the same leverage path.

To see more clearly how the model produces a sort of herding behavior, suppose that the market excess return $r(t) = \hat{r} > 0$ is constant and positive for a long time. In this case (52) can be solved using integrating factors to yield the characteristic equation

$$\xi(t) = \xi(0)e^{-ct} + \frac{(a - (a - b)u)\hat{r}}{c}(1 - e^{-ct}). \tag{54}$$

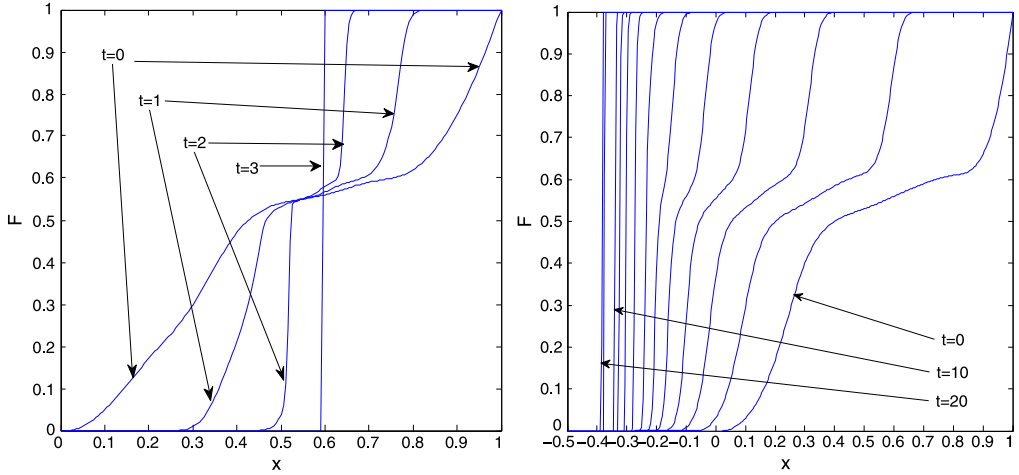


Fig. 7. Solutions to (49)–(50) using the numerical scheme detailed in Appendix B. Panel A. Positive returns, with parameter values $\hat{r} = 0.04$, $a = 8$, $b = 1$, $c = 0.3$. By time $t = 2$, the distribution contains two shocks, which combine by $t = 3$ to form one stable shock that converges to $x = \frac{(a+b)\hat{r}}{2c} = 0.6$. Panel B. Negative returns, with $\hat{r} = -0.01$, $a = 15$, $b = 8$, $c = 0.3$. Analysis of the characteristics guarantees a shock as soon as $F > 0$ at $x = -0.266$, which first happens near $t = 6$. By $t = 20$, the entire population mass is contained in a single shock that converges asymptotically to $x = \frac{(a+b)\hat{r}}{2c} = -0.383$.

From this equation (or from the logic of the previous paragraph), we see that all the characteristics move towards each other as time increases. Assuming that the mass of the distribution is initially contained within a finite interval $[X_l, X_u]$ on the x -axis, we conclude that each of the characteristics emanating from $[X_l, X_u]$ at $t = 0$ must eventually hit the inexhaustible set of characteristics that emanate from $x < X_l$ for which $F = 0$ or that emanate from $x > X_u$ for which $F = 1$. In other words, we converge to the solution for a single shock where $F = 0$ below the shock and $F = 1$ above the shock. By the Rankine–Hugoniot condition (46) and our expression for the flux H in (50), we know that this shock, $x = s(t)$, satisfies

$$\frac{ds}{dt} = \frac{1}{2}(a + b)\hat{r} - cs(t).$$

Solving this ODE for $t \geq \tau$ given a generic initial condition $s(\tau)$ yields

$$s(t) = \frac{1}{c} \left(\frac{1}{2}(a + b)\hat{r}(1 - e^{-c(t-\tau)}) + cs(\tau)e^{-c(t-\tau)} \right),$$

and so we see that, regardless of the value of $s(\tau)$, the shock exponentially converges to the long position $\hat{x} = \frac{(a+b)\hat{r}}{2c} > 0$. In other words, all the investors herd themselves towards the common positive leverage value \hat{x} . Panel A of Fig. 7 illustrates the evolution of the leverage distribution. Parameters are chosen so that \hat{x} is rather moderate at 0.6.

Behavior is even more interesting when we have constant negative returns, $r(t) = \hat{r} < 0$ for a long time. In this case the characteristics are

$$\xi(t) = \xi(0)e^{-ct} + \frac{(b + (a - b)u)\hat{r}}{c}(1 - e^{-ct}), \tag{55}$$

so starting with positive leverage $x = \xi(0)$ leads to rapid deleveraging $\frac{d\xi}{dt} < 0$ and, as noted earlier, the more highly leveraged managers deleverage faster. Thus we have a “race to the bottom”

that again typically involves shock waves. As illustrated in panel B of Fig. 7, the argument above tells us that, when $A = \mathbb{R}$, all managers will converge to the short position $\hat{x} = \frac{(a+b)\hat{r}}{2c} < 0$. When short-selling is prohibited (or prohibitively expensive), then managers will pile up at $x = 0$; that is, they all will exit the financial market until returns once again become positive.

Qualitative features of the foregoing analysis remain valid even when the market excess return r varies over time. Once a shock forms, it persists: the managers involved initially and those who join later will continue to clump together, and move in sympathy with shifts in r . This is a consequence of the entropy condition in the previous section, which stated that characteristics can only enter a shock; they cannot emanate from a shock as time evolves. According to the model, long spells of positive r will lead to herding at high leverage and a spell of negative r will lead to the herd moving in the opposite direction, dramatic deleveraging.

One can imagine extensions of the model that could capture other interesting features of financial markets, such as idiosyncratic returns and behavior across managers that might counterbalance herding. In a model complementary to ours, Friedman and Abraham [12] consider a finite population of heterogeneous managers and endogenize $r(t)$ (and the risk cost parameter c), but de-emphasize relative performance considerations. Their specification seems too complicated for analytic treatment, but their simulations, like ours, exhibit impressive bubbles and crashes.

5. Discussion

Much of this paper can be summed up in the following recipe for model construction:

- Take any symmetric two-player game $g(x, y)$ with a continuous action space $A \subset \mathbb{R}$.
- Extend to a population game $\phi(x)$ by averaging $g(x, y)$ over the distribution $F(t, y)$ of all opponents' actions.
- Impose gradient dynamics: each player adjusts his or her own action $x \in A$ to move up the payoff gradient at a rate proportional to the slope.
- Use techniques borrowed from fluid dynamics (and suitably extended) to characterize solutions to the resulting partial differential equations (PDEs).

We showed that these PDEs have classical solutions when g is a smooth function of x , and that the solution can be written explicitly when g is quadratic in x . Examples include several famous games in economics as well as strategy mixtures in any symmetric 2×2 game. On the other hand, we showed that for games g with discontinuities or kinks, one can get nonlinear PDEs with solutions that involve shock waves. After reviewing some techniques for addressing such complications, we analyzed a population of fund managers who interact in a financial market. There we saw how concern for relative performance can lead to a form of herding behavior, including shocks and, when returns become negative, a race to deleverage.

The assumption of gradient dynamics can be relaxed somewhat. When the velocity field V is proportional to, rather than exactly equal to, the payoff gradient ϕ_x , we can retain most analytic results by rescaling time t . Less obviously, the literature on differential inclusions or cone fields (e.g., Aubin and Celina [3], and Smale [31]) can be extended to the present setting. Apparently most qualitative results for gradient dynamics continue to hold as long as the velocity field V is commensurate with ϕ_x , in the sense that their ratio, $\frac{V}{\phi_x}$, will always fall between two positive constants.

The recipe and techniques presented here can be extended to wider classes of models. The easiest extension is to action spaces $A \subset \mathbb{R}$ that consist of several closed intervals, not just one. More ambitiously, the action space A could be a compact subset of \mathbb{R}^n . In a Hotelling model, for example, if the players choose price as well as location along a circle or line segment, then $n = 2$. Also, the underlying game g could be asymmetric, or involve more than two players, resulting in a population game with two or more strategically distinct player populations. For example, a population of buyers might interact with a population of sellers, each with a continuous action space. These models can lead to PDEs with more than one spatial dimension or to systems of PDEs that must be solved simultaneously. Fortunately the mathematics literature contains many results about such PDEs, opening a vast set of possible applications in biology and the social sciences.

Conversely, gradient dynamics PDE models present interesting new mathematical questions. Balance equations have shocks that correspond to discontinuities in their solutions, and there are other classes of PDEs whose solutions have discontinuous spatial derivatives at shocks, but not discontinuities in the solution itself; see Crandall and Lions [6] and Crandall, Evans, and Lions [5]. Both of these types of PDEs have specific structures that correspond to specific, but distinct, types of problems in the physical sciences. However, the economic model for preemption and Bertrand pricing in Subsection 4.2 leads to a gradient dynamics PDE, (38), that has aspects of both of the structures previously only seen separately. This makes this equation unique and unstudied. When characteristics cross, it is unclear if this corresponds to discontinuities in F or in F_x (although simulations suggest that the answer is F_x).⁹ Nor is uniqueness or even existence established for the solution of this equation after shocks form. Thus the study of gradient dynamics can provide a wide variety of interesting economic, biological, and social science models, as well as unique mathematical frontiers to explore.

Appendix A. Proofs

A.1. Proof of Theorem 1: Classical solutions for smooth g

Theorem 1. *Let $A = \mathbb{R}$ and let $g : A \times A \rightarrow \mathbb{R}$ have bounded second and third partial derivatives in its first argument. Also, let the initial density $f^0(x)$ have a bounded first derivative. Then neither the solution, $f(t, x)$, nor the partial derivative of the solution, $f_x(t, x)$, to the integro-partial differential equation defined by (2) and (4) can become unbounded in finite time.*

Proof. Given our assumptions on g , we can define B to be a bound on the absolute value of g_{xx} and the absolute value of g_{xxx} . Also, since f_x^0 is bounded and $\int_A f^0(x) dx = 1$, we have that f^0 is bounded. We first establish that f is bounded in finite time, and then we use that to establish that f_x is bounded in finite time.

⁹ Recall that we have, in general, required that $g(x, y)$ be absolutely continuous in x for this paper. The only $g(x, y)$ function that has not been absolutely continuous is the $g(x, y)$ for Bertrand pricing. This poses no difficulties when the distribution $F(t, x)$ stays smooth, as was the case in Subsection 4.2 prior to the shock formation. Should the shocks be in F_x , the fact that $g(x, y)$ is not continuous still poses no problem. However, should the shocks be in F , we would require a more careful analysis if the x value of the shock were ever to equal the x value where $g(x, y)$ is discontinuous. In that case, the exact nature of the discontinuity in g would have a significant effect on the solution. See Ostrov [24] for a discussion about how some discontinuities in the flux function H can affect solutions.

Combining (13) and (2), we have that along any characteristic

$$f(t, \xi(t)) = f^0(x^0) e^{-\int_0^t \int_A g_{xx}(\xi(s), y) f(s, y) dy ds},$$

so, since $|g_{xx}| \leq B$ and $\int_A f(s, y) dy = 1$, we have our bound for f :

$$|f(t, \xi(t))| \leq |f^0(x^0)| e^{Bt}.$$

To obtain the bound on f_x we need to use the PDE for f_x instead of f , which we obtain by differentiating (4) with respect to x and then applying the product rule to obtain

$$(f_x)_t + \tilde{V}(f_x)_x = -\tilde{V}_{xx} f - 2\tilde{V}_x f_x, \tag{56}$$

where the unknown function $\tilde{V}(t, x)$ is defined by $\tilde{V}(t, x) = V(x, F(t, \cdot)) = \int_A g_x(x, y) dF(t, y)$. The characteristic curves for f_x are the same as the curves for f , of course. Therefore, we still have that $\frac{d\xi}{dt} = \tilde{V}$ and so, by the chain rule, $\frac{df_x(t, \xi(t))}{dt}$ equals the left hand side of (56), which, after substituting, yields

$$\begin{aligned} \frac{df_x(t, \xi(t))}{dt} &= -\tilde{V}_{xx}(t, \xi(t)) f(t, \xi(t)) - 2\tilde{V}_x(t, \xi(t)) f_x(t, \xi(t)) \\ &= -\left[\int_A g_{xxx}(\xi(t), y) f(t, y) dy \right] f(t, \xi(t)) \\ &\quad - 2 \left[\int_A g_{xx}(\xi(t), y) f(t, y) dy \right] f_x(t, \xi(t)). \end{aligned}$$

Now we apply to this equation our bounds for the derivatives of g along with our obtained bound for f :

$$\left| \frac{df_x(t, \xi(t))}{dt} \right| \leq B |f^0(x^0)| e^{Bt} + 2B |f_x(t, \xi(t))|. \tag{57}$$

Consider the case where the three quantities in (57) contained within the absolute value signs are positive. In this case, we subtract the $2B f_x$ term from both sides, multiply by the integrating factor e^{-2Bt} , and then apply the product rule to obtain

$$\frac{d[e^{-2Bt} f_x(t, \xi(t))]}{dt} \leq B f^0(x^0) e^{-Bt}.$$

Then, integrating, we have that

$$f_x(t, \xi(t)) \leq e^{2Bt} (f_x^0(x^0) + f^0(x^0)(1 - e^{-Bt})) \leq e^{2Bt} (f_x^0(x^0) + f^0(x^0)). \tag{58}$$

Applying a similar process to the cases where various quantities within the absolute value signs are negative, we obtain the following modification of (58):

$$|f_x(t, \xi(t))| \leq e^{2Bt} (|f_x^0(x^0)| + |f^0(x^0)|). \tag{59}$$

Since f^0 and f_x^0 are bounded, we see from (59) that f_x is also bounded in finite time.

Therefore, shocks (that is, discontinuities) cannot form in either $F(t, x)$ or $f(t, x)$ for these integro-partial differential equations. \square

A.2. Proof of Theorem 2: Solution for quadratic g

Theorem 2. Let

$$g(x, y) = ax + \frac{b}{2}x^2 + cxy + h(y) \tag{60}$$

or let the velocity field take the form

$$V(x, F(t, \cdot)) = a + bx + c\mu^F(t), \tag{61}$$

where $x \in A = \mathbb{R}$, $t \geq 0$, and a, b , and c are real constants. Then, given an initial probability density, $f^0(x)$, with compact support, the gradient dynamics PDE (4) has the following unique solution:

$$f(t, x) = e^{-bt} f^0\left(\mu^F(0) + \left(x + \frac{a}{b+c}\right)e^{-bt} - \left(\mu^F(0) + \frac{a}{b+c}\right)e^{ct}\right) \tag{62}$$

and the mean is

$$\mu^F(t) = \left(\mu^F(0) + \frac{a}{b+c}\right)e^{(b+c)t} - \frac{a}{b+c}. \tag{63}$$

This solution is a classical solution if f^0 is differentiable. That is, the partial derivatives in the gradient dynamics PDE (4) are defined and satisfy this PDE at every point (t, x) when $t > 0$.

Proof. We need to show that if $A = \mathbb{R}$, then Eq. (4) with

$$V(x, F(t, \cdot)) = a + bx + c\mu^F(t)$$

yields (62). Eq. (4) in this case is

$$f_t = -[(a + bx + c\mu^F(t))f]_x.$$

Integrating x from $-\infty$ to y and then integrating y from $-\infty$ to ∞ gives

$$\frac{d}{dt} \int_{-\infty}^{\infty} \int_{-\infty}^y f(t, x) dx dy = -a - (b+c)\mu^F(t). \tag{64}$$

Applying integration by parts to the outer integral in the left hand side of this equation gives

$$\frac{d}{dt} \int_{-\infty}^{\infty} \int_{-\infty}^y f(t, x) dx dy = - \lim_{y \rightarrow -\infty} yF_t(t, y) + \lim_{y \rightarrow \infty} yF_t(t, y) - \frac{d}{dt} \left(\int_{-\infty}^{\infty} yf(t, y) dy \right).$$

Given the compact support of $f^0(x)$ and the fact that information travels along characteristics, whose slope cannot become infinite in finite time, we have that $f(t, x)$ has compact support in x at any given time. Since the bounds of this support are continuous in time, we also have that $F_t(t, y)$ has compact support in y . Therefore, we can simplify

$$- \lim_{y \rightarrow -\infty} yF_t(t, y) + \lim_{y \rightarrow \infty} yF_t(t, y) - \frac{d}{dt} \left(\int_{-\infty}^{\infty} yf(t, y) dy \right) = 0 + 0 - \frac{d\mu^F(t)}{dt}.$$

Substituting this into Eq. (64) yields $\frac{d\mu^F(t)}{dt} = a + (b+c)\mu^F(t)$, which we can solve by separation of variables to obtain

$$\mu^F(t) = \left(\mu^F(0) + \frac{a}{b+c} \right) e^{(b+c)t} - \frac{a}{b+c}.$$

With $\mu^F(t)$ now known explicitly, we have that the velocity field $V(x, F(t, \cdot)) = v(t, x)$, where v is an explicit function of t and x . This means we can apply (11), the characteristic equation for $\xi(t)$ from Subsection 3.2:

$$\frac{d\xi}{dt} = v(t, \xi) = a + b\xi + c \left[\left(\mu^F(0) + \frac{a}{b+c} \right) e^{(b+c)t} - \frac{a}{b+c} \right].$$

Since this is a linear ODE, we subtract $b\xi$ from both sides, multiply by the integrating factor e^{-bt} , apply the product rule to the left hand side and integrate, yielding

$$\xi(0) = \mu^F(0) + \left(\xi(t) + \frac{a}{b+c} \right) e^{-bt} - \left(\mu^F(0) + \frac{a}{b+c} \right) e^{ct}.$$

Finally, we apply (13), the evolution of f along a characteristic, and the fact that $x = \xi(t)$ to obtain our desired result:

$$f(t, x) = e^{-bt} f^0 \left(\mu^F(0) + \left(x + \frac{a}{b+c} \right) e^{-bt} - \left(\mu^F(0) + \frac{a}{b+c} \right) e^{ct} \right).$$

With (62) and (63) established, we can directly compute the derivatives in the gradient dynamics PDE (4) at all (t, x) points when $t > 0$, as long as f^0 is differentiable. Confirming that these derivatives satisfy the PDE, we have that the solution is classical. \square

A.3. Generalization of Theorem 2 results

We obtain a solution to the continuity equation when $A = \mathbb{R}$, f has compact support, and $g(x, y) = ax + \frac{b}{2}x^2 + cxy + dxy^2 + h(y)$. Then $g_x(x, y) = a + bx + cy + dy^2$ and $V(x, F(t, \cdot)) = \int_A g_x(x, y) f(t, y) dy$, so the continuity equation is of the form

$$[f(t, x)]_t = - \left[\int_{-\infty}^{\infty} (a + bx + cy + dy^2) f(t, y) dy f(t, x) \right]_x. \tag{65}$$

If we define $\mu_1(t)$ and $\mu_2(t)$ to be the first and second moments of x :

$$\begin{aligned} \mu_1(t) &= \mu^F(t) = \int_{-\infty}^{\infty} x f(t, x) dx \\ \mu_2(t) &= \int_{-\infty}^{\infty} x^2 f(t, x) dx, \end{aligned}$$

the continuity equation (65) can be rewritten in the form

$$[f(t, x)]_t = - \left[(a + bx + c\mu_1(t) + d\mu_2(t)) f(t, x) \right]_x. \tag{66}$$

Recalling that f has compact support, if we integrate (66) from $-\infty$ to y in x and then integrate from $-\infty$ to ∞ in y , we have that

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{\infty} \int_{-\infty}^y f(t, x) dx dy &= - \int_{-\infty}^{\infty} (a + by + c\mu_1(t) + d\mu_2(t)) f(t, y) dy \\ &= -a - (b + c)\mu_1(t) - d\mu_2(t). \end{aligned} \tag{67}$$

Applying integration by parts to the exterior integral on the left hand side of (67) and using the compact support of f yields

$$\frac{d}{dt} \int_{-\infty}^{\infty} \int_{-\infty}^y f(t, x) dx dy = - \frac{d}{dt} \int_{-\infty}^{\infty} y f(t, y) dy = -\mu_1'(t) \tag{68}$$

and combining (67) and (68) gives the ODE

$$\mu_1'(t) = a + (b + c)\mu_1(t) + d\mu_2(t). \tag{69}$$

On the other hand, if we integrate (66) from $-\infty$ to y in x , multiply by y , and then integrate from $-\infty$ to ∞ in y we have that

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{\infty} y \int_{-\infty}^y f(t, x) dx dy &= - \int_{-\infty}^{\infty} y(a + by + c\mu_1(t) + d\mu_2(t)) f(t, y) dy \\ &= -a\mu_1(t) - b\mu_2(t) - c(\mu_1(t))^2 - d\mu_1(t)\mu_2(t). \end{aligned} \tag{70}$$

Applying integration by parts to the exterior integral on the left hand side of (70) yields

$$\frac{d}{dt} \int_{-\infty}^{\infty} y \int_{-\infty}^y f(t, x) dx dy = - \frac{d}{dt} \int_{-\infty}^{\infty} \frac{y^2}{2} f(t, y) dy = -\frac{1}{2}\mu_2'(t) \tag{71}$$

so combining (70) and (71) gives the ODE

$$\mu_2'(t) = 2a\mu_1(t) + 2b\mu_2(t) + 2c(\mu_1(t))^2 + 2d\mu_1(t)\mu_2(t). \tag{72}$$

We can solve this system of two autonomous ODEs, (69) and (72), numerically and then insert the now known functions for $\mu_1(t)$ and $\mu_2(t)$ back into the continuity equation (66). With $\mu_1(t)$ and $\mu_2(t)$ inserted, (66) is now reduced to a time-varying exogenous landscape and can be easily solved numerically through the methods provided in Subsection [Appendix B](#).

Remark. This method can be extended to any $g(x, y)$ that is the sum of terms of the form $cx^n y^m$ where c is constant, n is 0, 1 or 2 and m is a nonnegative integer. To do this, one determines and then solves the ODE system for the first M moments where M is the largest value of m appearing in the terms. So, for example, $g(x, y) = 7x^2 - 15y^4 + 23xy^6$ can be handled by solving the six ODEs for the first six moments, but $g(x, y) = 7x^3$ cannot be accommodated since the power of x is greater than 2.

Appendix B. Numerical techniques

This appendix presents the numerical techniques used to generate the figures in Subsections 2.3, 3.4, and 4.4 of this paper. Readers should bear in mind, as detailed in Subsections 3.2

and 4.1, that the solution's dependence on initial conditions evolves strictly along the characteristic curves. They should also recall, as detailed in Section 4, how shocks form in solutions to the balance equations.

The task is to numerically simulate the solution, $F(t, x)$, of (5), the gradient dynamics PDE

$$F_t = -V F_x. \quad (73)$$

Finite difference methods were used in this paper, which means F is approximated over the grid points of a rectangular mesh in the (t, x) plane. Let Δt represent the interval between successive times in the mesh, and Δx represent the interval between successive x values in the mesh. That is, the grid points that make up the mesh are $(t, x) = (n\Delta t, i\Delta x)$, where $n = 0, 1, 2, \dots$ and $i = p, p+1, p+2, \dots, q$, given the action space is of the form $A = [p\Delta x, q\Delta x]$. As is standard in the field of numerical analysis of PDEs, we use F_i^n to denote the numerical approximation to $F(n\Delta t, i\Delta x)$, the actual solution at these grid points. That is,

$$F_i^n \approx F(n\Delta t, i\Delta x).$$

In the three subsections below, we consider numerical techniques for three different cases. The first case is where the adjustment velocity function $V(x, F(t, \cdot)) = c$, a constant. Many of the important issues in numerical simulations of PDE arise within this simple context. By understanding these issues, it becomes clear how to work with the second case, where V is continuous; that is, the case where there are no shocks in the interior of the action space A . This encompasses the examples from Subsection 2.3 where $g(x, y) = |x - y|^{0.5}$ and where $g(x, y) = \pm(x - y)^2$ in $V(x, F(t, \cdot)) = \int_A g_x(x, y)f(t, y) dy$. It also encompasses the Keynes' beauty contest example from Subsection 3.4, where $V(x, F(t, \cdot)) = c\mu^F(t) - x$. The third case covers balance equations, which permit shocks in the interior of A . It encompasses the remaining examples, namely, Subsection 2.3 where $g(x, y) = \pm|x - y|$ and Financial Market Shocks from Subsection 4.4.

For all the examples in Subsections 2.3 and 3.4, we use $\Delta x = 0.02$, but the examples of Subsection 4.4 require finer resolution, and here we use $\Delta x = 0.005$.

B.1. $V = c$, a constant

Many useful insights can be gleaned from the simple case where $V = c$, a constant:

$$F_t = -cF_x.$$

Approximating F_t is simple enough:

$$F_t(n\Delta t, i\Delta x) \approx \frac{F(n\Delta t + \Delta t, i\Delta x) - F(n\Delta t, i\Delta x)}{\Delta t} \approx \frac{F_i^{n+1} - F_i^n}{\Delta t},$$

and so

$$F_i^{n+1} \approx F_i^n - \Delta t c F_x(n\Delta t, i\Delta x). \quad (74)$$

The question of how to approximate F_x is more complicated. A numerical scheme is called *unstable* if it magnifies small approximation errors with each successive time step. In computer simulations, unstable schemes exhibit small oscillations that quickly grow out of control and crash the simulation. While the most obvious choice for approximating F_x is the centered difference

$$F_x(n\Delta t, i\Delta x) \approx \frac{F_{i+1}^n - F_{i-1}^n}{2\Delta x},$$

using this in (74) yields an unstable numerical scheme. We therefore consider using the forward difference

$$F_x(n\Delta t, i\Delta x) \approx \frac{F_{i+1}^n - F_i^n}{\Delta x}$$

or the backward difference

$$F_x(n\Delta t, i\Delta x) \approx \frac{F_i^n - F_{i-1}^n}{\Delta x}$$

to approximate F_x .

Only one of these can lead to a stable scheme, and its stability further depends on Δt being sufficiently small, as we now explain. Take the mesh point $((n + 1)\Delta t, i\Delta x)$, that is, the point where we want to find F_i^{n+1} . Now trace the characteristic that passes through this point backwards in time by a single time step, Δt , which brings us back to time $n\Delta t$. There are three possibilities for the x value of the characteristic at this time. It might fall between $i\Delta x$ and $(i + 1)\Delta x$, the two points on which the forward difference is based; it might fall between $(i - 1)\Delta x$ and $i\Delta x$, the two points on which the backward difference is based; or, if Δt is too large (and $c \neq 0$), it will fall outside of the region between $(i - 1)\Delta x$ and $(i + 1)\Delta x$.

In the last of these three possibilities, the numerical scheme is guaranteed to be unstable. To prevent this, we must make certain to choose a time step, Δt , that is small enough so that

$$\Delta t \leq \frac{\Delta x}{|c|}.$$

This is called the CFL (Courant, Friedrichs and Lewy) stability condition. It is a necessary, but not sufficient, condition for stability. With the first of the three possibilities, the forward difference is stable; the backward difference is not. With the second possibility, this is reversed: the backward difference is stable; the forward difference is not. In other words, for stability, the characteristic must travel between the two points used to approximate F_x . Since we must switch between the forward or backward difference depending on the sign of c , we can use the following numerical scheme:

$$F_i^{n+1} = F_i^n - \frac{\Delta t}{\Delta x} (\max\{c, 0\}(F_i^n - F_{i-1}^n) + \min\{c, 0\}(F_{i+1}^n - F_i^n)).$$

This scheme is called an *upwind* scheme, because it follows the direction that characteristic information flows. It is stable provided the CFL condition is satisfied.

B.2. V is continuous (F has no shocks in the interior of A)

We can expand this constant velocity upwind scheme to the case where V is a continuous function in t and x in a relatively straightforward manner. Reflecting our definition of F_i^n , we define V_i^n to be the numerical approximation to V at the grid points:

$$V_i^n \approx V(i\Delta x, F(n\Delta t, \cdot)).$$

We can determine V_i^n by numerically approximating $V(x, F(t, \cdot)) = \int_A g_x(x, y)f(t, y) dy$ when $g(x, y) = |x - y|^{0.5}$ and when $g(x, y) = \pm(x - y)^2$ from Subsection 2.3 and by numerically approximating $V(x, F(t, \cdot)) = c\mu^F(t) - x$ in Keynes' beauty contest from Subsection 3.4. In both cases the numerical approximation uses our knowledge of F_j^n for $j = p, p + 1, \dots, q$, that is, the approximation of F at the current time $n\Delta t$ at all x grid points. Since V is continuous,

the fact that we are approximating V at $(n\Delta t, i\Delta x)$, instead of $((n+1)\Delta t, i\Delta x)$, makes no real difference.

Therefore, our upwind numerical scheme now becomes

$$F_i^{n+1} = F_i^n - \frac{(\Delta t)_n}{\Delta x} (\max\{V_i^n, 0\}(F_i^n - F_{i-1}^n) + \min\{V_i^n, 0\}(F_{i+1}^n - F_i^n)),$$

where the CFL condition is

$$(\Delta t)_n \leq \frac{\Delta x}{\max_{j \in \{p, p+1, \dots, q\}} |V_j^n|}.$$

Note that Δt can change at each new time step, so it is now indexed by n . Also note that at each new time step, Δt must simultaneously satisfy the CFL condition at all x grid points.

B.3. Balance equations where V may not be continuous and F may have shocks

Our final examples, namely Subsection 2.3 when $g(x, y) = \pm|x - y|$ and Subsection 4.4 on Financial Market Shocks, correspond to the balance equation form (49):

$$F_t(t, x) + [H(t, x, F(t, x))]_x = Q(F(t, x)).$$

In these examples, we have a potential problem in that F is, generically, a discontinuous function, which also makes V discontinuous. However, we can still create a stable, accurate upwind scheme by expanding the material in Section 13.5 of LeVeque [19] that details upwind schemes for scalar conservation laws. Specifically, we use

$$F_i^{n+1} = F_i^n - \frac{(\Delta t)_n}{\Delta x} (\mathcal{H}(t, x, F_i^n, F_{i+1}^n) - \mathcal{H}(t, x, F_{i-1}^n, F_i^n)) + (\Delta t)_n Q(F_i^n),$$

where we define the function $\mathcal{H}(t, x, F_1, F_2) = \min_{F \in [F_1, F_2]} H(t, x, F)$. The CFL condition in this case is

$$(\Delta t)_n \leq \frac{\Delta x}{\max_{j \in \{p, p+1, \dots, q\}} |H_F(n\Delta t, j\Delta x, F_j^n)|}.$$

This upwind scheme belongs to a class of schemes called “monotone” schemes. The numerical solution obtained from any monotone scheme was proved by Crandall and Majda [7] to converge to the correct unique solution of the balance equation as $\Delta x \rightarrow 0$ (which also forces $\Delta t \rightarrow 0$ by the CFL condition). As detailed in Section 4, this correct unique balance equation solution is the solution where all shocks obey both the entropy condition and the Rankine–Hugoniot jump condition.

References

- [1] S. Anderson, J. Goeree, C. Holt, Noisy directional learning and the logit equilibrium, *Scand. J. Econ.* 106 (3) (2004) 581–602.
- [2] K. Arrow, L. Hurwicz, Stability of the gradient process in n -person games, *J. Soc. Ind. Appl. Math.* 8 (1960) 280–294.
- [3] J.-P. Aubin, A. Cellina, *Differential Inclusions*, Springer-Verlag, 1984.
- [4] J. Chevalier, G. Ellison, Risk taking by mutual funds as a response to incentives, *J. Polit. Economy* 105 (6) (1997) 1167–1200.
- [5] M.G. Crandall, L.C. Evans, P.-L. Lions, Some properties of viscosity solutions of Hamilton–Jacobi equations, *Trans. Amer. Math. Soc.* 282 (2) (1984) 487–502.

- [6] M.G. Crandall, P.-L. Lions, Viscosity solutions of Hamilton–Jacobi equations, *Trans. Amer. Math. Soc.* 277 (1) (1983) 1–42.
- [7] M.G. Crandall, A. Majda, Monotone difference approximations for scalar conservation laws, *Math. Comput.* 34 (149) (1980) 1–21.
- [8] R. Cressman, J. Hofbauer, Measure dynamics on a one-dimensional continuous trait space: Theoretical foundations for adaptive dynamics, *Theor. Popul. Biol.* 67 (2005) 47–59.
- [9] C.M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, second edition, *Grundlehren Math. Wiss. (Fundamental Principles of Mathematical Sciences)*, vol. 325, Springer-Verlag, Berlin, 2005, xx+626 pp., ISBN 978-3-540-25452-2; 3-540-25452-8.
- [10] D. Del Guercio, P.A. Tkac, The determinants of the flow of funds of managed portfolios: mutual funds versus pension funds, *J. Finan. Quant. Anal.* 37 (2002) 523–557.
- [11] L.C. Evans, *Partial Differential Equations*, *Grad. Stud. Math.*, vol. 19, American Mathematical Society, Providence, RI, ISBN 0-8218-0772-2, 1998, xviii+662 pp.
- [12] D. Friedman, R. Abraham, Bubbles and crashes: Gradient dynamics in financial markets, *J. Econ. Dynam. Control* 33 (4) (2009) 922–937.
- [13] D. Friedman, D. Ostrov, Gradient dynamics in population games: Some basic results, *J. Math. Econ.* 46 (5) (2010) 691–707.
- [14] R. Guenther, J. Lee, *Partial Differential Equations of Mathematical Physics and Integral Equations*, Dover Publications, Inc., Mineola, NY, ISBN 0-486-68889-5, 1996, xii+562 pp.
- [15] S. Hart, A. Mas-Colell, Regret-based continuous-time dynamics, *Games Econ. Behav.* 45 (2003) 375–394.
- [16] J. Hofbauer, J. Oechssler, F. Riedel, Brown–von Neumann–Nash dynamics: The continuous strategy case, *Games Econ. Behav.* 65 (2009) 406–429.
- [17] J. Karceski, Returns-chasing behavior, mutual funds, and beta’s death, *J. Finan. Quant. Anal.* 37 (2002) 559–594.
- [18] J.M. Keynes, *The General Theory of Employment, Interest and Money*, Macmillan Cambridge University Press, ISBN 0-15-634711-3, 1936, Chapter 12 V.
- [19] R.J. LeVeque, *Numerical Methods for Conservation Laws*, second edition, *Lectures Math. ETH Zürich*, Birkhäuser Verlag, Basel, ISBN 3-7643-2723-5, 2005, 232 pp.
- [20] D. Luenberger, Complete stability of noncooperative games, *J. Optim. Theory Appl.* 25 (4) (1978) 485–505.
- [21] J. Maynard Smith, *Evolution and the Theory of Games*, Cambridge University Press, New York, 1982.
- [22] R. Nagel, Unraveling in guessing games — An experimental study, *Amer. Econ. Rev.* 95 (1995) 1313–1326.
- [23] J. Oechssler, F. Riedel, On the dynamic foundation of evolutionary stability in continuous models, *J. Econ. Theory* 107 (2002) 223–252.
- [24] D. Ostrov, Solutions of Hamilton–Jacobi equations and scalar conservation laws with discontinuous space–time dependence, *J. Differential Equations* 182 (1) (2002) 51–77.
- [25] A. Rogers, Beating your neighbor to the Berry patch, unpublished manuscript, University of Utah Anthropology Department, July 1995.
- [26] Paul M. Romer, Increasing returns and long-run growth, *J. Polit. Economy* 94 (5) (1986) 1002–1037.
- [27] M. Ruijgrok, T.W. Ruijgrok, Replicator dynamics with mutations for games with a continuous strategy space, unpublished manuscript, Mathematics Institute, University of Utrecht, May 2005.
- [28] W. Sandholm, *Population Games and Evolutionary Dynamics*, MIT Press, ISBN 978-0-262-19587-4, 2011.
- [29] H. Simon, *Models of Man*, John Wiley, New York, 1957.
- [30] E.R. Sirri, P. Tufano, Costly search and mutual fund flows, *J. Finance* 53 (1998) 575–603.
- [31] S. Smale, Exchange processes with price adjustment, *J. Math. Econ.* 3 (1976) 211–226.
- [32] J. Smoller, *Shock Waves and Reaction–Diffusion Equations*, second edition, *Grundlehren Math. Wiss. (Fundamental Principles of Mathematical Sciences)*, vol. 258, Springer-Verlag, New York, ISBN 0-387-94259-9, 1994, xxiv+632 pp.