Preemption Games: Theory and Experiment

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Abstract

Several impatient investors with private costs C_i face an indivisible irreversible investment opportunity whose value V is governed by geometric Brownian motion. The first investor i to seize the opportunity receives the entire payoff, $V - C_i$. We characterize the symmetric Bayesian Nash equilibrium for this game. A laboratory experiment confirms the model's main qualitative predictions: competition drastically lowers the value at which investment occurs; usually the lowest cost investor preempts the other investors; observed investment patterns in competition (unlike monopoly) are quite insensitive to changes in the Brownian parameters. Support is more qualified for the prediction that markups decline with cost.

Keywords: Preemption, Incomplete Information, Irreversible Investment, Laboratory Experiment.

JEL codes: C73, C92, D82, G13

El Mutún, perhaps the world's largest iron ore deposit, was opened to private investors in the 1980s but, due to the high cost of developing the remote Bolivian site, there were no takers for two decades. In late 2005, spurred by rising commodity prices, the Brazilian company EBX finally seized the opportunity, preempting rivals based in China and India (Andrew J. Barden and Adrianna Arai, 2006). Numerous similar examples can be found in the annals of mining and oil companies (Raymond F. Mikesell et. al., 1971).

The strategic problem is as old as humanity: the value of a grove of figs fluctuates as the fruit ripens (or is attacked by worms) and the first band of hunter-gatherers to seize it eats better than neighboring bands. High tech firms today face similar issues when they introduce a new product. Delaying introduction might allow the product niche to expand, but new substitutes or changes in standards might shrink the niche. A rival could preempt the niche or it could vanish entirely due to a disruptive new technology (Clayton M. Christensen, 1997). Similar considerations apply to retailers deciding when to open a "big box" outlet in a market too small to handle more than one store, and also apply to academic researchers investigating a hot new topic.

In this paper we study such situations both theoretically and empirically. We formalize them as preemption games, using standard simplifications to put the strategic issues into sharp focus.[1] In our games, the opportunity is available to a known number $n+1$ of investors; it has a publicly observed value V that evolves according to geometric Brownian motion with known parameters; each investor has a privately known avoidable cost of investing; and the first mover preempts the entire value V.

Our model builds on an active literature reviewed in Marcel Boyer et. al. (2004) that studies preemption contests for investments with option values. Unlike most of this literature (e.g., Steven Grenadier, 2002; Helen Weeds, 2002; Romain Bouis et al, 2006) firms in our model are uncertain of their rivals' costs. A tradeoff for this added piece of realism is that unlike some of this literature but like much classic auction theory, preemption is complete; the winner takes all of the returns from investment. A separate literature considers preemption in very different environments related to R&D (e.g., Jennifer F. Reinganum, 1981; Drew Fudenberg and Jean Tirole, 1985; Heidrun C. Hoppe and Ulrich Lehmann-Grube, 2005) and market entry (e.g., Dan Levin and James Peck, 2003).

Lambrecht and Perraudin (2003) is our direct predecessor. Like us, they draw on real options theory to investigate preemption of a stochastic investment opportunity by competitors whose costs are private information. Our theoretical results extend theirs by covering more than two competitors, and by relaxing a restrictive technical assumption. Our model, unlike theirs, is explicitly rooted in auction theory as well as in real options theory. Indeed, special cases of our model include Dutch auctions as well as deferral options.

Our theoretical contribution appears in Section I. We characterize the symmetric Bayesian-Nash equilibrium of the preemption game with an arbitrary number of players. Players' BNE strategies take the form of a threshold value at which the opportunity is seized immediately. The mapping from realized cost to equilibrium threshold is characterized by an ordinary differential equation as well as by a new recursion equation. We show that the auction and real option special cases lead

[1]Reality, of course, is always more complex. In the El Mutún example, Bolivia's newly-elected government shut down EBX's operations in 2006, citing environmental and other concerns, and in late 2007 signed a 40 year concession contract with the Indian firm Jindal. Other parts of the story are more consistent with our model. The number of serious rivals was always reasonably clear. The true costs of the Brazilian and Indian firms (and their Chinese rival, Shandong) were, in no small part, private information due to confidential subsidies arranged by their own governments as well as confidential understandings with the Bolivian government. The firms faced the hazard that the Bolivian government might renationalize El Mutún, or declare it a protected national park, before any of them could seize the opportunity (Barden and Arai, 2006; Patrick J. McDonnell, 2008).

to useful bounds on the BNE strategies.

Section II describes a laboratory experiment informed by the theory, using software created expressly for the purpose. It presents the main treatments—Competition (triopoly) vs Monopoly, and High vs Low Brownian parameters—and obtains four testable hypotheses. Section III explains other aspects of the laboratory implementation.

Section IV presents the results. The first three hypotheses fare quite well: the triopoly market structure leads to much lower markups (threshold value less cost) than the monopoly structure; the Brownian parameters have a major impact in the predicted direction in Monopoly but (again as predicted in BNE) have negligible impact in Competition; and the lowest cost investor indeed is far more likely to preempt than her rivals. The evidence is mixed on the last hypothesis: at the low cost end of the scale, investors' markups indeed tend to decline in cost, but the predicted relationship breaks down at higher costs. However, these departures from prediction turn out to have very little impact on subjects' actual earnings.

Following a concluding discussion, Appendix A provides proofs and mathematical details. Other appendices available online compute the equilibrium of a restricted version of the preemption game, discuss the lesser-known econometric techniques, report supplementary data analysis, and reproduce instructions to subjects. Our theoretical contribution incorporates and extends the PhD dissertation of Steven T. Anderson (2003). A companion paper, Ryan D. Oprea, Daniel Friedman and Steven T. Anderson (forthcoming) describes a related laboratory experiment concerning the monopoly ($n = 0$) case only.

I Theoretical Results

This section analyzes two situations. In the first, called monopoly, a single investor i has sole access to an investment opportunity. In the second, called competition, two or more investors with private information concerning their own costs have access to the same opportunity, and the first to seize it renders it unavailable to the others.

A Monopoly

An investor i with discount rate[2] $\rho > 0$ can launch a project whenever she chooses by sinking a given cost $C_i > 0$. The present value V of the project evolves via geometric Brownian motion with drift parameter $\alpha < \rho$ and volatility parameter $\sigma > 0$:

$$dV = \alpha V dt + \sigma V dz, \tag{1}$$

where z is the standard Wiener process. That is, the value follows a continuous time random walk in which the appreciation rate has mean α and standard deviation σ per unit time. At times $t \geq 0$ prior to launching the project, the investor observes $V(t)$. If she invests at time t, then she obtains payoff $[V(t) - C_i]\, e^{-\rho t}$. The project is irreversible and generates no other payoffs. Thus, the task is to choose the investment time so as to maximize the expected payoff.

The solution goes back to Claude Henry (1974) and has been widely known since Robert L. McDonald and Daniel R. Siegel (1986); see Chapter 5 of Dixit and Pindyck (1994) for a detailed exposition. The optimal policy takes the form: wait until $V(t)$ hits the threshold

$$V_M(C_i) = (1 + w)C_i, \tag{2}$$

then launch immediately. Note that the threshold is proportional to cost, and that the wait option premium $w \geq 0$ is an algebraic function of the volatility, drift and discount parameters σ, α and ρ. Specifically,

$$w = \frac{1}{\beta - 1}, \texttt{ where } \beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left[\frac{\alpha}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}} > 1. \tag{3}$$

B Competition

Now consider the case that each investor has $n \geq 1$ rivals. All investors $i = 1, 2, ..., n + 1$ have access to the same investment opportunity, whose value V again evolves according to geometric Brownian motion (1). Each investor i again knows her own cost C_i, but doesn't know the other investors' costs C_j, $j \neq i$. She regards them as independent draws from a cumulative distribution function, $H(C)$, with a positive continuous density function $h(C)$ on support $[C_L, C_U]$. The first investor to launch, say at time $t_i > 0$, obtains payoff $[V(t) - C_i]e^{-\rho t_i}$ and the other investors obtain

[2] Recall that the discount rate reflects pure time preference, the expiration hazard and possibly risk aversion. As explained in the next section, the laboratory experiment focuses on the expiration hazard, sometimes informally referred to as "preemption by Nature."

zero payoff. All this is common knowledge. The resulting preemption game is denoted $\Gamma(\beta, n, H)$.

We show that the preemption game has a unique symmetric Bayesian Nash Equilibrium (BNE). It is characterized by an increasing function $V^*(C_i)$ that maps the investor's cost into a threshold value, above which she immediately invests. We now sketch the derivation and offer some intuition; details and proofs appear in Appendix A.

Using notation m, V_o and $\hat{C}$ defined below, the expected discounted payoff $E[V(t) - C_i]e^{-\rho t_i}$ can be written out as the following objective function:

$$F(m|C_i, n) = [V^*(m) - C_i] \left[\frac{V_o}{V^*(m)}\right]^\beta \left[\frac{1 - H(m)}{1 - H(\hat{C})}\right]^n . \tag{4}$$

The choice variable in (4) is $m \in [C_L, C_U]$, interpreted as the cost-type that the investor chooses as her potential "masquerade".[3]

The first factor in the objective function (4) is simply the profit (or "markup") $[V^*(m) - C_i]$ obtained at the time of successful investment. The second factor, $[V_o/V^*(m)]^\beta$, accounts for the time cost of delaying investment and the expiration hazard, given that V_o is the current value of the investment project. Appendix A.1 shows that the monopolist's value function consists of only these first two factors. Appendix A.2 notes that with competitors ($n > 0$), the restriction $\rho > \alpha$ can be relaxed and consequently (4) is valid for $\beta \geq 0$, while of course (2-3) is valid only for $\beta > 1$.

The third and final factor, $\left[(1 - H(m))/(1 - H(\hat{C}))\right]^n$, is the probability that the n rivals all have higher costs (and therefore will not preempt), conditioned on the fact that none of them has already invested. That conditioning is reflected in the denominator. Let $\hat{V} \geq V_o$ be the "highest peak" so far achieved by the random walk. Then $\hat{C}$ is the corresponding cost, i.e., $\hat{V} = V^*(\hat{C})$. Since the preemption game is over as soon as the first investor moves, it turns out that the BNE threshold strategy is independent of $\hat{V}, \hat{C}$ and V_o within the relevant range.

The key to obtaining the BNE is the best response (or "truthtelling") property that investor i maximizes (4) at $m = C_i$. The associated first-order condition can be expressed as the following ordinary differential equation (ODE):

$$V^{*\prime}(C_i) = \frac{[V^*(C_i) - C_i]\, V^*(C_i)}{V^*(C_i) - \beta\,[V^*(C_i) - C_i]} \times \frac{nh(C_i)}{[1 - H(C_i)]}. \tag{5}$$

[3] Of course, the threshold value itself is the natural choice variable but, as in auction theory, it turns out that m is more convenient since $V^*(.)$ is invertible.

For reasons explained in the next subsection, we also impose the boundary condition

$$V^*(C_U) = C_U. \tag{6}$$

This boundary value problem has a unique solution V^* that characterizes the symmetric BNE threshold for our preemption game. Appendix A.2 shows that it can also be expressed as a conditional expectation and that it satisfies

$$V^*(C) = C + \int_C^{C_U} \left[\frac{V^*(C)}{V^*(y)}\right]^\beta \left[\frac{1-H(y)}{1-H(C)}\right]^n dy. \tag{7}$$

Theorem 1. *Let the cumulative distribution function H have a continuous density h with full support $[C_L, C_U]$, where $0 < C_L < C_U < \infty$. Let it be common knowledge among all investors $i = 1, ..., n+1$ that i's investment cost C_i is an independent random variable with distribution H that is observed only by investor i. Then*

1. *for any $\beta \geq 0$, boundary value problem (5-6) has a unique solution $V^* : [C_L, C_U] \to R$,*
2. *a function V^* satisfies the recursion equation (7) iff it solves the boundary problem (5-6), and*
3. *the premption game $\Gamma[\beta, n, H]$ has a symmetric Bayesian-Nash equilibrium in which each investor i's threshold is V^* evaluated at realized cost C_i.*

Lambrecht and Perraudin (2003) use slightly different methods to derive an ODE that is consistent with Equation (5) for the special case that (a) there are only two investors (duopoly), and (b) H has an increasing modified hazard rate $C_i\,[h(C_i)]\,/\,[1-H(C_i)]$. Their methods rule out asymmetric BNE. Our method considers only symmetric BNE but has several compensating advantages. It covers $n+1 > 2$ investors, allows arbitrary hazard rates, connects explicitly to auction theory, and leads to a streamlined and unified analysis of useful special cases.

C Special Cases, Intuition, and Bounds

Consider again the special case $n = 0$, monopoly. Since n is a factor in the numerator in (5), it might seem at first that the expression is identically zero. However, the last paragraph of Appendix A.1 notes that in this case the denominator $V^*(C_i) - \beta[V^*(C_i) - C_i] = 0$ and that this last expression characterizes the monopoly solution.

Appendix A.4 shows that, at any realized cost C_i, the monopoly solution $V_M(C_i)$ is an upper bound on the competitive solution $V^*(C_i)$. The intuition is simple but revealing. In monopoly, an investor increases her threshold up to the point that the greater profit margin just balances the greater threat of "preemption by Nature," i.e., the expiration hazard. With competition, the investor must also consider preemption by other investors, so she finds the balance at a lower threshold.

The special case $\beta = 0$ is also instructive. Here the middle factor of the objective function (4) disappears, yielding

$$F(m|C_i, n) = [V^*(m) - C_i] \left[\frac{1 - H(m)}{1 - H\left(\hat{C}\right)} \right]^n . \tag{8}$$

Apart from the denominator, which disappears in the first order condition, this is a familiar objective function from auction theory. Thus the ODE (5) collapses to the ODE for bid functions in a reverse Dutch (or reverse first price) auction:

$$V^{*\prime}(C_i) = [V^*(C_i) - C_i] \times \frac{nh(C_i)}{[1 - H(C_i)]}. \tag{9}$$

The boundary condition (6), of course, remains unchanged. Hence, with $\beta = 0$, we obtain an isomorphism between threshold values in preemption games and bids in auctions.

The "auction" solution $\bar{V}(C_i)$ to boundary value problem (9, 6) is another upper bound on the solution to the more general boundary value problem (5, 6). The intuition again is that the investor chooses the threshold to balance greed (a larger profit margin) and fear (of preemption) at the margin. Completely eliminating a preemption threat (in this case from Nature, by setting $\beta = 0$, e.g., by setting $\alpha > 0 = \rho$) sets an upper bound on the balance point.

The BNE threshold $V^*(C_i)$ also has a natural lower bound, given by the classic Marshallian (i.e., zero net present value) investment rule $V^0(C_i) = C_i$. Recall that the boundary condition (6) imposes this rule, but only at the highest possible cost C_U. The intuition for the boundary condition is compelling. An investor with the highest possible cost faces Bertrand competition: every rival will find it profitable to undercut any positive markup she might seek.

These bounds and special cases are summarized in the following

Theorem 2. *Under the hypotheses of Theorem 1, the BNE threshold $V^*(C_i)$ is an increasing, continuously differentiable function bounded below by the Marshallian threshold function $V^0(C_i) = C_i$ and bounded above by the monopoly threshold $V_M(C_i)$. It is also bounded above by the solution $\bar{V}(C_i)$ to (9, 6) and is tangent to $\bar{V}(C_i)$ at the upper endpoint C_U.*

Finally, consider special cases of the cost distribution. If H has a constant density (i.e., is uniform) or a continuous decreasing density, then it can be shown (see Appendix A.5) that the BNE threshold function V^* is concave.

Our experiment employs the uniform distribution H on $[C_L, C_U]$. In this case, (5) reduces to

$$V^{*\prime}(C_i) = \frac{[V^*(C_i) - C_i]\, V^*(C_i)}{V^*(C_i) - \beta\,[V^*(C_i) - C_i]} \times \frac{n}{C_U - C_i}. \tag{10}$$

In the special case $\beta = 0$, we then get the well-known ODE from Vickrey (1961)

$$V^{*\prime}(C_i) = [V^*(C_i) - C_i] \times \frac{n}{C_U - C_i}, \tag{11}$$

with analytic solution

$$V^{**}(C_i) = \frac{nC_i + C_U}{n+1}. \tag{12}$$

Corollary 1. *Let H be the uniform distribution on $[C_L, C_U]$, let n be an integer ≥ 1, and let $V^{**}(C_i)$ be given by Equation (12). For $\beta > 0$, the Bayesian-Nash equilibrium threshold function $V^*(C_i)$, the solution to (10, 6) on $[C_L, C_U]$, is concave, bounded above by V^{**}, tangent to V^{**} at C_U, and bounded below by the Marshallian threshold function $V^0(C_i) = C_i$.*

Thus when the cost distribution is uniform, an upper bound (and often a good approximation) of the BNE threshold V^* is the function

$$V^U(C_i) = \min\{V^{**}(C_i); V_M(C_i)\} \tag{13}$$

The monopolist threshold binds in (13) at lower cost realizations for high values of β and low values of n.

D Numerical Example

Figure 1 shows the numerical solution V^* to the boundary value problem (10, 6) for triopoly ($n = 2$), given the uniform cost distribution on $[C_L, C_U] = [50, 80]$, for $\beta = 2.25$ and $\beta = 3.00$. The first β value is referred to as "High" because it generates higher monopoly thresholds (via option premium $w = 0.8$) than the second β value, which has $w = 0.5$ and is referred to as "Low." Panel (a) includes the monopoly thresholds V_M. Panel (b) shows the BNE threshold functions at finer resolution, together with the Marshallian lower bound $V^0(C_i) = C_i$ and the Vickrey upper bound $V^{**}(C_i)$. One can see that the two BNE threshold functions lie close together and that, as claimed

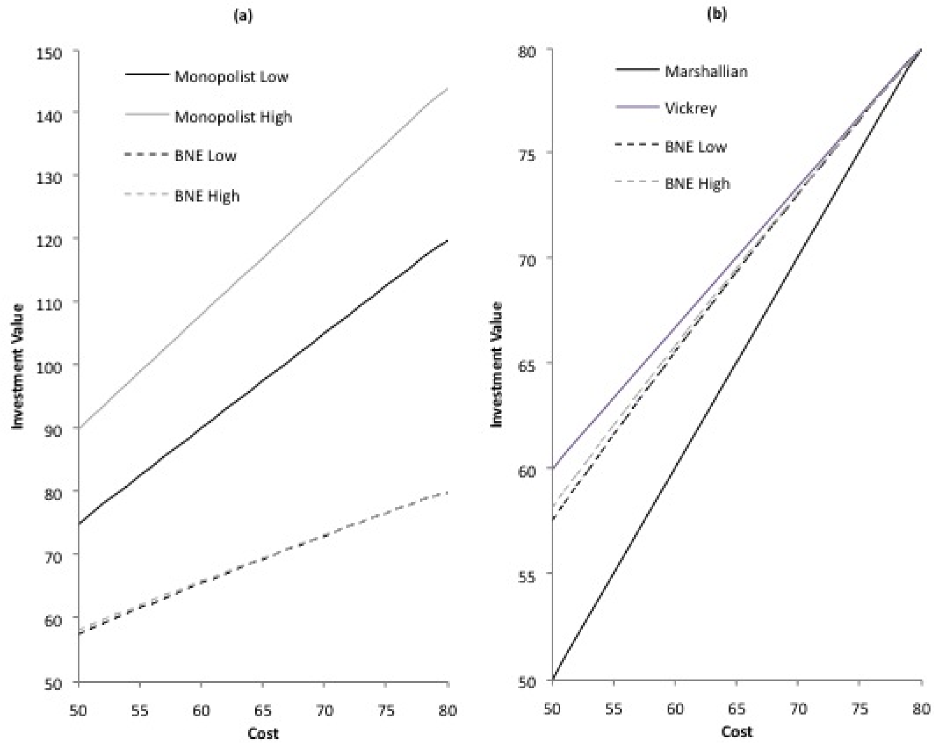

Figure 1: BNE threshold values for triopolies with costs drawn independently from the uniform distribution on [50, 80]. "High" refers to the case $\beta = 2.25$ with option premium $w = 0.8$, and "Low" refers to the case $\beta = 3.00$ with option premium $w = 0.5$.

in Theorem 2 and Corollary 1, both are tangent to the Vickrey bound at the upper endpoint.

E Discrete Approximations

Brownian motion is an idealization. Our experiment uses a close binomial approximation of the continuous time process. Specifically, it has a fixed time interval $\Delta t = 0.00\bar{3}$ minutes (i.e., 200 milliseconds) for each discrete step of the value path, and three binomial parameters:

- the step size $\eta > 0$ of the proportional change in value, i.e., the current value V becomes either $(1+\eta)V$ or $(1-\eta)V$ at the next step;
- the uptick probability $p \in (0, 1)$, i.e., the probability that the next step is to $(1+\eta)V$ rather than to $(1-\eta)V$; and
- the expiration probability $q \in (0, 1)$, i.e., the probability that the current step is the last, and the opportunity disappears.

	Parameters			
Treatment	η	**p**	**q**	**Replications**
Low	0.03	0.524	0.007	36 subjects
High	0.03	0.513	0.003	36 subjects

Table 1: Binomial parameters and number of subjects studied in each treatment.

As noted in Appendix A.6, the discrete binomial parameters $(p, q, \eta, \Delta t)$ map into the Brownian parameters (α, σ, ρ). The Brownian parameters in turn have a sufficient statistic, β, which determines he monopoly ($n = 0$) threshold and the BNE ($n > 0$) threshold. Thus, our choices of n and the binomial parameters in the laboratory allow us to manipulate the equilibrium thresholds and thus to test the theory's predictive power.

II Treatments and Hypotheses

Two binary treatment variables, Parameters and Structure, allow us to test the major predictions of the model. Parameters fixes the time step at $\Delta t = 0.00\bar{3}$ (in minutes) and the step size at $\eta = 0.03$. Parameters = Low is shorthand for $p = 0.524$ and $q = 0.007$, corresponding to $\beta \approx 3.0$ and option premium $w \approx 0.5$, as in the numerical example of the previous section. Likewise, Parameters = High is shorthand for $p = 0.513$ and $q = 0.003$, corresponding to $\beta \approx 2.25$ and $w \approx 0.8$. These configurations differ considerably from each other, yet both yield value paths "in the money" often enough, and jagged enough, to maintain subjects' interest. Table 1 summarizes this treatment variable for easy reference.

The second treatment variable is Structure. When Structure = Monopoly, the subjects make investment decisions with no rivals ($n = 0$) and therefore no risk of preemption by another investor. When Structure = Competition, the subjects compete in triopolies ($n = 2$). In each Competition period the subjects are randomly reassigned to one of three or four separate markets, each with three investors.

Each period each subject's cost is drawn independently from $U[50, 80]$, the uniform distribution with support [50, 80]. Sessions begin with a 10 period Monopoly block, MonopolyI, continue with 25 periods of Competition, and end with MonopolyII, another 10 period Monopoly block. The data analysis focuses on the MonopolyI (hereafter simply Monopoly) and Competition block. Online Appendix D analyses MonopolyII data and obtains parallel (but somewhat more diffuse) results.

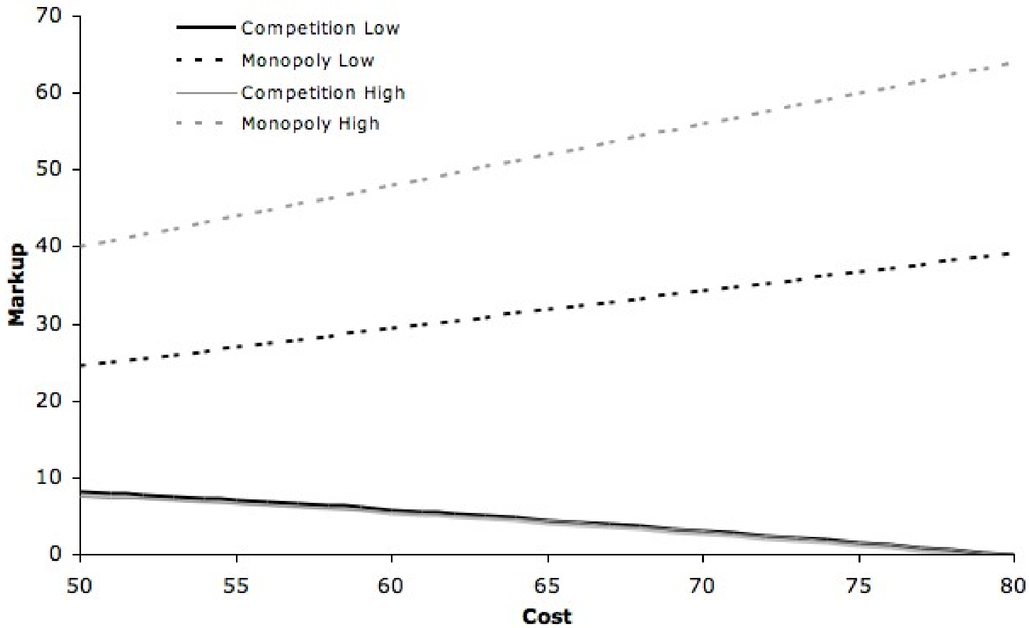

Figure 2: Predicted markups by Treatment.

Figure 2 plots markups, i.e., threshold value less cost. For both High and Low parameters, the dotted lines represent Monopoly markup, $V_M^*(C_i) - C_i$, and the solid lines represent BNE markups in Competition, $V^*(C_i) - C_i$. The figure shows that the Competition markups are everywhere much lower than the corresponding Monopoly markups. Our first hypothesis is that the markups observed in the experiment will have the same ordering as these theoretical constructs.

Hypothesis 1. *(Structure.) Markups chosen by subjects in the Monopoly treatments significantly exceed markups chosen in the Competition treatments.*

Another striking aspect of Figure 2 is that the Monopoly line for the High parameter vector is far above the corresponding Low line, while under Competition the two lines are very close together. The second hypothesis is that the experimental data will reflect this aspect of the theory.

Hypothesis 2. *(Parameters.) Markups chosen in the High Monopoly treatment significantly exceed those in Low Monopoly, while chosen markups have the same distribution in the High Competition data as in Low Competition.*

Recall that the symmetric BNE strategy $V^*(C_i)$ is increasing in cost C_i. A direct implication is that the investment opportunity is always seized by the lowest cost investor. Allowing for some behavioral noise, we obtain the following efficient sorting hypothesis:

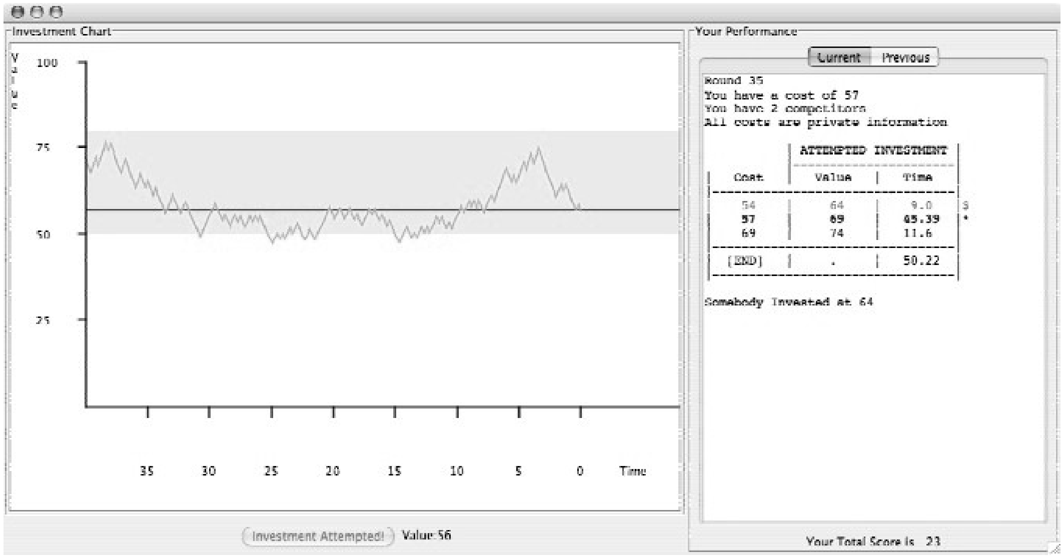

Figure 3: Subject screen under Competition at the end of a period.

Hypothesis 3. *(Efficiency.) Under Competition, the most efficient (lowest cost) subject is the one most likely to preempt the others.*

A final observation from Figure 2 is that, under our parameters, Competition markups decrease in costs. The reason is that the BNE threshold slope $V^{*\prime}(C_i) < 1$; indeed the approximate threshold function (13) has a slope of 2/3, so the theoretical markup slope in Competition is close to -1/3.

Hypothesis 4. *(Monotonicity.) Under Competition, observed markups are decreasing in cost.*

A complementary hypothesis, drawn from real options theory and observable in Figure 2, is that Monopoly markups are instead increasing in cost. This is a central hypothesis tested by Oprea, Friedman and Anderson (2009) who find strong empirical support.

III Implementation

Experiments were conducted using customized software called InvestmentTiming. Figure 3 shows the user interface. The lightly shaded band (colored blue on subjects' screens) indicates the cost range, $[50, 80]$, which was held constant throughout the session and announced publicly. The horizontal line (colored red on subjects' screens) represents the subject's own cost that period; its status as private information was also announced publicly.

The current value of investment, $V(t)$, was represented by a jagged line (colored green on subjects' screens) that evolved from the right, as on a seismograph. During each period the value line was initialized at 50 (the lower bound of the cost distribution) and evolved from there according to the binomial parameter vector, High or Low, chosen for that session. The vertical axis rescaled if the value line ever rose out of the displayed bounds.

Subjects were not allowed to invest when the value line was below their own cost, to prevent negative earnings, nor could they invest after the random ending time. At all other times, subjects could attempt to invest by tapping the space bar at their computer terminal.

In the Monopoly treatment, an investment attempt prior to period end was always successful, immediately netting a subject $V(t) - C_i$ points. Subjects in the Competition treatment were not told whether or when their competitors invested until after the period was over.[4] This semi-strategy method gives us access to more data, while still giving subjects the real-time choice experience that we feel helps them adapt to the stochastic environment.[5]

After the period ended, subjects were told the time at which each subject in their group attempted investment, the value at which they attempted investment, the costs of each competitor, and the resulting profits: $V(t)-C_j$ to the subject who invested first, and zero to the others. In the Monopoly treatment, of course, there were no other subjects in the group.

All cost draws, value sequences and period endings were made only once for each parameter set, and were repeated in all sessions for that treatment. In one session under High parameters, a software malfunction during period 30 (towards the end of the Competition block) lead to 4 missing periods which have been dropped from the dataset.

Subjects were given instructions in two parts, reproduced in online Appendix E. First, instructions pertaining to monopoly periods were distributed and read aloud. Binomial parameters and exchange rates for the session were written on a white board, and were pointed out several times before play began. Following six unpaid practice periods, we conducted the first Monopoly block. Then the second part of the instructions, pertaining to Competition, were distributed and read aloud. The instructions were identical across sessions; only the parameters written on the white board varied.

Experiments were conducted at the University of California, Santa Cruz, using inexperienced undergraduate subjects recruited from a large online database of volunteers. Subjects each were paid a $5 showup fee. They earned 5 cents per point in Monopoly periods and (to maintain a comparable payout rate in triopoly) 15 cents per point in Competition periods. Low sessions averaged one hour

[4] As Remark A1 shows, this design choice does not alter the BNE.

[5] Implementing the full strategy method would require us to constrain the strategy space, e.g., to a choice of threshold $V(C)$, excluding, a priori, non-stationary and other sorts of strategies that subjects might otherwise use.

and thirty minutes while High sessions averaged close to two hours. On average, subjects in the Low treatment earned \$15.21 and subjects in the High treatment earned \$12.57.

IV Results

Subjects frequently failed to invest prior to the random end of the period. Efforts to invest at particularly high values are therefore censored in many periods while efforts to invest at lower values are more likely to be observed. Consequently the sample of observed investment decisions is downward biased.

We employ two different techniques to correct the bias. The first is to construct product limit (PL) estimates of the empirical cumulative distribution function of markups. As explained in online Appendix C, the PL procedure uses all data (including all instances of censoring) to construct maximum likelihood non-parametric estimates. The estimates are graphed for each treatment in Figure 4.

The top panel of Figure 4 suggests that investment behavior is considerably different under Competition than under Monopoly. For example, the median markups (where the graph crosses the horizontal line at 0.5) are about 5 or 6 for both Competition treatments versus about 12 for Low Monopoly and about 21 for High Monopoly. The observed ordering seems consistent with the first two hypotheses.

The bottom panels of Figure 4 compare observed distributions (solid lines) with the predicted distributions (dashed lines), obtained as PL estimates on artificial data generated by applying theoretical functions V^* and V_M^* to the realized cost draws. In the Competition panel, the predictions for High and Low parameters are very close together. The CDFs for observed markups for High and Low parameters are also close to each other, and only slightly more diffuse than the predictions. In the Monopoly panel, the predictions for High parameters are about 15-20 points higher (i.e., to the right of) those for Low parameters at each percentile. Above the 30th percentile, the CDFs for observed markups have the same ordering and about the same spacing above the 80th percentile, but for the most part they fall well below (i.e., to the left of) the theoretical predictions.

Our second approach to the censoring problem is to examine a data subsample in which, for exogenous reasons, censoring is rare. Theory guides the subsampling: in BNE for Competition there is no censoring in periods in which the maximum, $\hat{v}$, of the value line is above 80. Likewise in the Monopoly prediction there is no censoring when $\hat{v}$ exceeds 120 in the Low treatment or 144

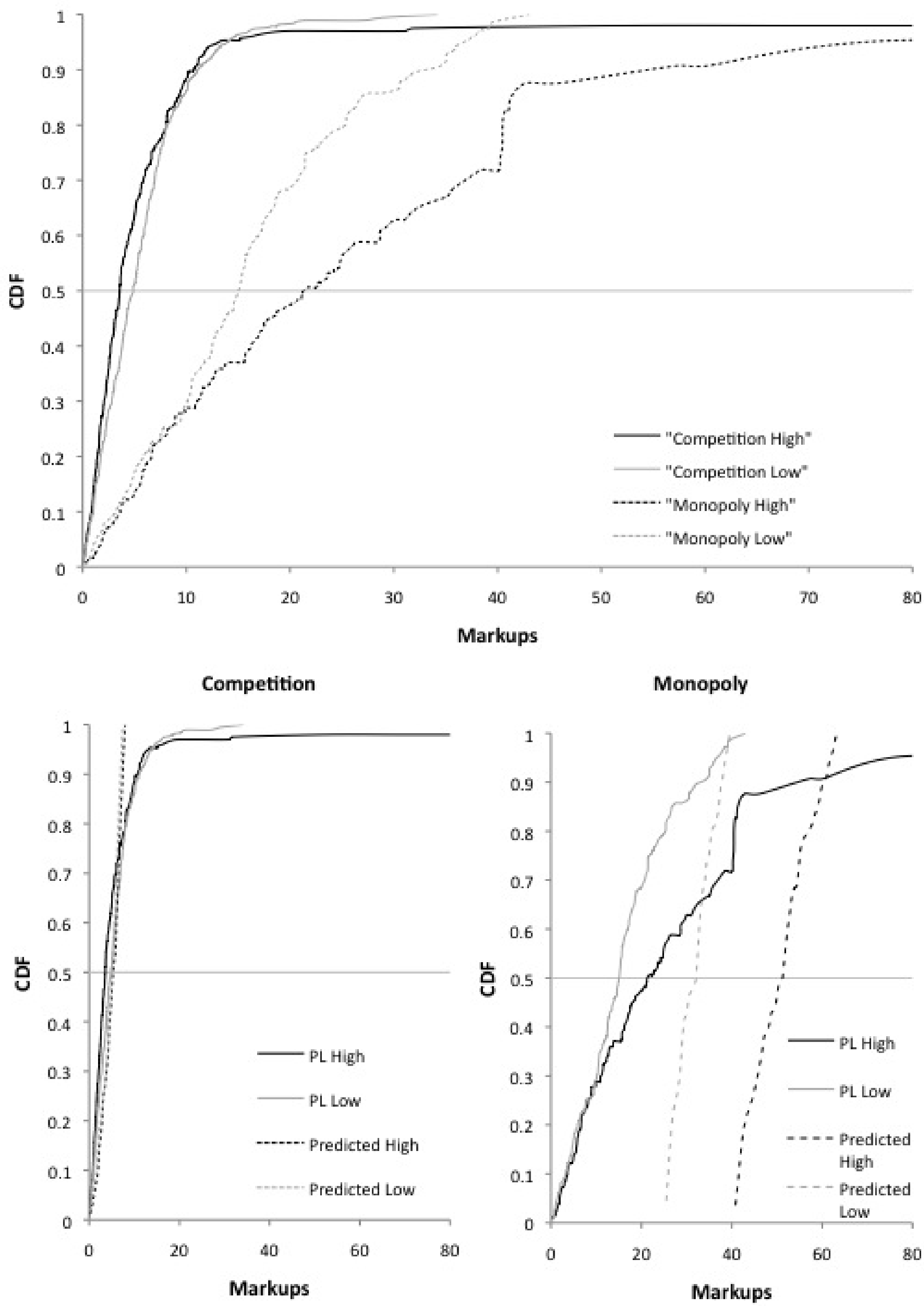

Figure 4: Product Limit Cumulative Distribution Functions.

in the High treatment. We refer to the sets of such periods as "feasible" samples, since in theory investment was feasible for all investors. Note that $\hat{v}$ is uncorrelated with cost and, because it is unknown to subjects ex ante, it should also be uncorrelated with the chosen markup.

The feasible samples turn out, in fact, to be nearly uncensored. In Competition periods, we observe nearly 97 percent of 660 decisions in the subsample (versus 85 percent of 1039 decisions overall) while in Monopoly periods we observe over 98 percent of 72 decisions (versus 61 percent of 519 overall). Therefore standard econometric tools can be used confidently on the feasible samples.

A Treatment Level Hypothesis Tests

The PL estimates in Figure 4 seem consistent with the first two hypotheses. To formalize the inferences, we estimate the following nested random effects regression on the feasible sample:[6]

$$V_{jit} - C_{jit} = \gamma + \psi High_j + \kappa Monopoly_t + \delta High_j \times Monopoly_t + \nu_j + u_i + \epsilon_{jit}. \tag{14}$$

The left hand side is the observed markup, $High_j$ is an indicator variable taking a value of 1 in High parameter set sessions and 0 otherwise, and $Monopoly_t$ is an indicator variable for Monopoly periods. The variable ν_j is a random effect on sessions and u_i is a random effect on individual subjects, both assumed to be normally distributed with a mean of zero. Together these account for within subject and within session correlations. Finally, ϵ_{jit} is a normally distributed, mean zero disturbance term.

Results are displayed in Table 2. Estimates of treatment effects are constructed from the coefficient estimates as follows: Competition Low = γ; Competition High = $\gamma+\psi$; Monopoly Low = $\gamma+\kappa$; and Monopoly High = $\gamma+\psi+\kappa+\delta$. Thus Hypothesis 1, that Monopoly markups exceed Competition markups, translates to $\gamma+\kappa > \gamma$ (or $\kappa > 0$) for Parameters = Low, and translates to $\gamma+\psi+\kappa+\delta > \gamma+\psi$ (or $\kappa+\delta > 0$) for Parameters = High. Table 2 confirms the Low case: κ is significantly greater than zero at the one percent level. A Wald test confirms the High case: $\kappa+\delta$ is greater than zero ($p = 0.000$).

Finding 1. *Consistent with Hypothesis 1, markups are significantly lower under Competition than under Monopoly given either Low or High Parameters.*

[6] Here we pool data from all treatments and make cross treatment comparisons. For consistency, we uniformly apply the most stringent criterion, $\hat{v} > 144$. Using the less stringent criteria where applicable (e.g., $\hat{v} > 80$ in Competition blocks) does not appreciably change the estimates and does not alter the results of the hypothesis tests reported below.

Variable	Coefficient	Estimate
Intercept	γ	5.929
		(1.542)***
High	ψ	-0.866
		(1.839)
Monopoly	κ	12.979
		(2.128)***
$High \times Monopoly$	δ	9.184
		(2.784)***

Table 2: Estimates (and standard errors) from nested random effects model (14). Notes: Variables *High* and *Monopoly* are dummies for the Parameter and Structure treatment variables. One, two and three stars designate significance at the ten percent, five percent and one percent levels.

The second hypothesis is that investment values are sensitive to binomial parameters under Monopoly but not under Competition. Equation (14) translates the hypothesis as $\gamma + \psi + \kappa + \delta > \gamma + \kappa$ (or $\psi + \delta > 0$) but $\gamma + \psi$ is no greater than γ (or ψ is insignificantly different from zero). A Wald test indicates that $\psi + \delta$ is indeed significantly larger than zero ($p = 0.019$). Table 2 reports that ψ is insignificantly different from zero. Thus:

Finding 2. *Consistent with Hypothesis 2, investment values in Monopoly periods are significantly larger under High parameters than Low parameters, while there is no significant difference in Competition periods.*

B Preemption, Markups and Cost in Competition

The third hypothesis is that the lowest cost investor (the efficient one) will usually preempt her rivals. Figure 5 shows the fraction of times that the preemptor has the lowest, middle and highest cost. In both treatments the lowest cost investor wins roughly 80 percent of the time while the highest cost investor wins about 4 percent of the time in the Low treatment and hardly ever in the High treatment. Equally important, the exceptional cases are mostly when costs are very close. On average, the difference between the middle and low cost draws is $(C_U - C_L)/(n + 2) = (80 - 50)/(3 + 1) = 7.5$ points but in periods when a second lowest cost subject preempts the lowest cost subject, the median difference is only 2.5. Likewise, when a highest cost subject preempts the lowest cost subject, the median difference is only 4, as compared to an *a priori* average of $2 \times 7.5 = 15$ points. To summarize,

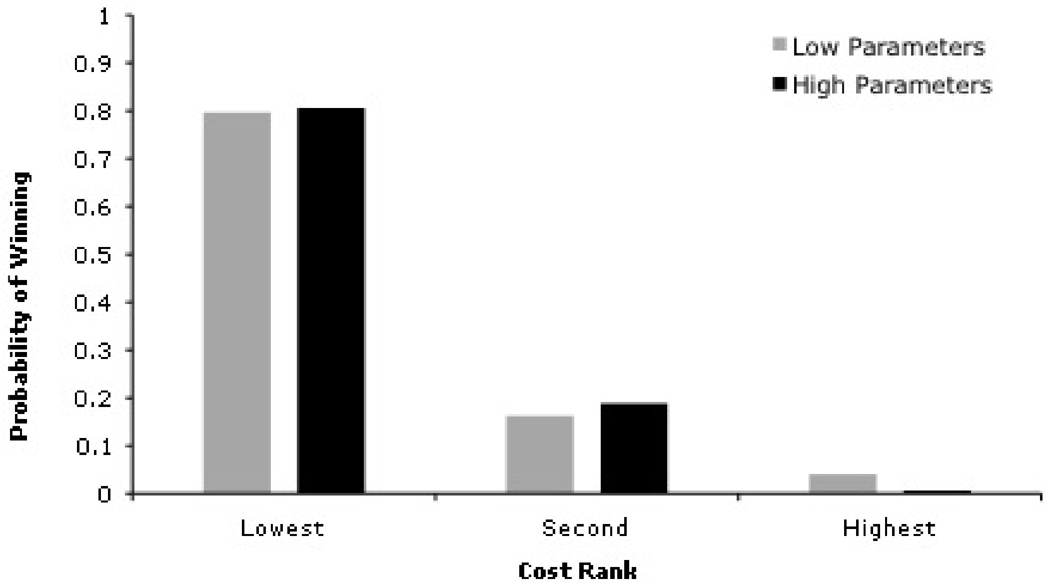

Figure 5: Probability of preempting both rivals as a function of cost rank. The fraction of cases in which the highest cost competitor successfully invests in the High treatment is 0.006, too small to be seen in the figure.

Finding 3. *Consistent with Hypothesis 3, the lowest cost investor usually preempts the other investors, and the highest cost investor rarely preempts the other investors.*

Hypothesis 4 predicts that under Competition, the target markup $M = V(C) - C$ is decreasing in the cost of investment C; indeed, the BNE functions decline almost linearly. In testing this hypothesis, we focus on the feasible subsample (periods in which $\hat{v} > 80$). Figure 6 shows the average markups by cost range for each treatment. The first impression is that there is no relation, and that markups are approximately constant. To check this impression, we estimate the following nested random effects regression

$$V_{it} - C_{it} = a + bC_{it} + \nu_j + u_i + \epsilon_{it} \tag{15}$$

where u_i and ν_j are random effects on subjects and session respectively (both assumed normal with mean zero) and ϵ is an error term distributed $N(0, \sigma_\epsilon^2)$. The column labelled (1) in Table 3 reports the cost coefficient estimates, which are negative but significantly smaller (in absolute value) than the predicted coefficients shown in the previous column. Indeed, one can see from the standard errors that these coefficient estimates are not significantly different from zero. Moreover, the intercept estimates are less than half of the BNE intercepts.

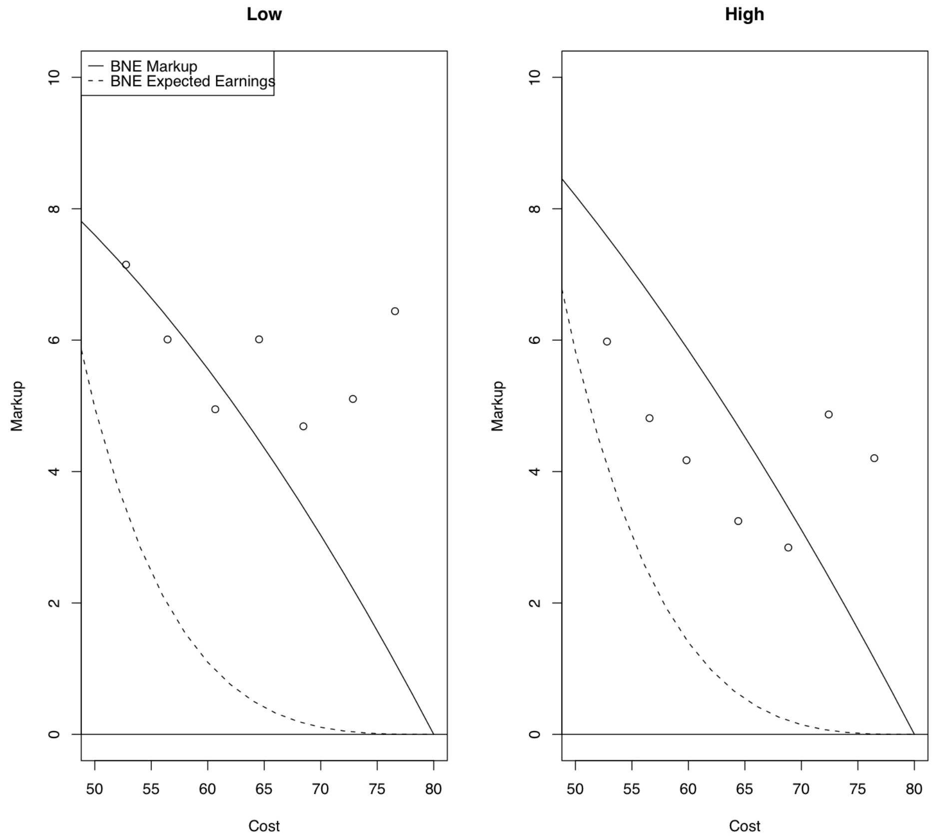

Figure 6: Average observed markups by cost interval are shown as open circles for the feasible subsample, $\hat{v} > 80$, of Competition periods. The BNE markups appear as solid lines, and the expected earnings from BNE play (given by Equation (4)) appear as dashed lines.

An alternative hypothesis is that subjects use cognitively simpler constant markup strategies. Online Appendix B characterizes the Nash Equilibrium (NE) in threshold strategies constrained to be constant across costs. In the NE of the constrained game, the slope coefficient is of course zero and the intercept turns out to be 7.15 (resp. 7.81) in the Low (resp. High) treatment. In column (1), under both treatments, the slope is insignificantly different from zero and 95 percent confidence intervals for the intercepts cover the NE values. Thus, column (1) is consistent with the constrained Nash equilibrium.

Before rejecting Hypothesis 4, however, one should recall that the incentives for playing BNE strategies are not constant across costs. Figure 6 includes a dotted line charting the expected earnings from BNE play according to Equation (4). Earnings drop rapidly, so deviations from the BNE strategy are cheap at higher costs. Could weak incentives at higher costs confound our statistical inference? A closer look at Figure 6 confirms that the BNE prediction does much

	BNE	**(1)** Overall	**(2)** $E(\pi) > 0.1$	**(3)** $E(\pi) > 1$	**(4)** $Weighted$
			Panel A: Low		
Estimates					
Intercept	20.65	8.148***	11.096***	21.4388**	14.392***
		(1.878)	(3.142)	(8.281)	(2.876)
Cost	-0.253	-0.0377	-0.0872	-0.2742*	-0.144***
		(0.0282)	(0.051)	(0.149)	(0.050)
Wald Tests					
Intercept = BNE		0.0000	0.003	0.924	0.030
Cost = BNE		0.0000	0.001	0.887	0.029
N		386	267	117	386
			Panel B: High		
Estimates					
Intercept	22.14	7.9283***	15.8773***	25.2618***	18.1915***
		(2.540)	(3.239)	(8.143)	(3.397)
Cost	-0.273	-0.0561	-0.192***	-0.359**	-0.229***
		(0.0391)	(0.053)	(0.144)	(0.0583)
Wald Tests					
Intercept = BNE		0.0000	0.129	0.703	0.246
Cost = BNE		0.0000	0.055	0.552	0.451
N		251	165	92	251

Table 3: Coefficient estimates (and standard errors) from nested random effects regressions (15) for the feasible sample, $\hat{v} > 80$. Notes: One, two and three stars designate significance at the ten percent, five percent and one percent levels. Wald Tests (e.g. "Intercept=BNE") report p-values from Wald test of the equality of estimated coefficients to the BNE coefficients. Columns 1-4 report results for the full feasible sample, for subsamples in which BNE play provides expected returns exceeding 0.1 and 1, and for a weighted regression in which error amplitude is assumed to be inversely proportional to expected return in BNE.

better in the lower cost part of the range, where expected earnings are far larger. Indeed, in the Low treatment, the open circles lie parallel to the prediction line over the range $E(\pi_{BNE}) > 1$ (or $C \leq 60$), and in the High treatment this relationship continues over the wider range $E(\pi_{BNE}) > 0.1$ (or $C \leq 70$).

To follow up, we rerun (15) on the subsample in which $E(\pi_{BNE}) > 0.1$, and report the results in column (2) of Table 3. As predicted in Hypothesis 4, the slope coefficient is indeed much more negative than in column (1), and is quite significant in the High treatment. In neither treatment can we reject the BNE hypothesis at the 5 percent level or better. In the more stringent subsample $E(\pi_{BNE}) > 1.0$ reported in column (3), the estimated slope coefficient is almost exactly the BNE value in the Low treatment, and is marginally significantly *more* negative than the BNE value in the High treatment.

To extend this line of thought, suppose that behavioral noise is inversely related to expected earnings. Specifically, assume $\epsilon \sim N(0, \sigma_W^2/E(\pi_{BNE}))$, i.e., the error term ϵ is heteroskedastic with a variance inversely proportional to expected earnings in BNE, and re-run (15) accordingly. Column (4) in Table 3 reports the results. In both treatments, there is a negative relationship between markups and costs that is economically and statistically significant. In the High treatment, both intercept and slope are indistinguishable from BNE levels by Wald tests. In the Low treatment the empirical intercept is lower than the BNE intercept and the slope term is somewhat less negative.

Finding 4. *Except at higher costs (where expected payoff is very small), the observed markups are consistent with BNE in the High Competition treatment, and they also decline in cost in the Low treatment. Overall, the payoff-weighted evidence supports Hypothesis 4.*

Finding 4 begs the question of how much money subjects leave on the table due to behavioral noise. Would subjects significantly improve earnings by reducing the noise? To put it another way, is BNE a better response to the empirical distribution of investment choices than subjects' actual strategies?

To answer such questions, we calculate the gains to BNE: what each subject would have earned had she unilaterally played the BNE throughout the experiment, less what she actually earned. Figure 7 shows that the median gain is quite small: about 1.24 points (or \$0.18) per session in Low and 1.52 points (or about \$0.22) in High. A Mann-Whitney test cannot distinguish these medians from zero ($p = 0.493$ in Low and $p = 0.1974$ in High). Overall, then, behavioral noise seems rather inexpensive.

Finding 5. *Against the empirical distribution of play, the differences between actual earnings and*

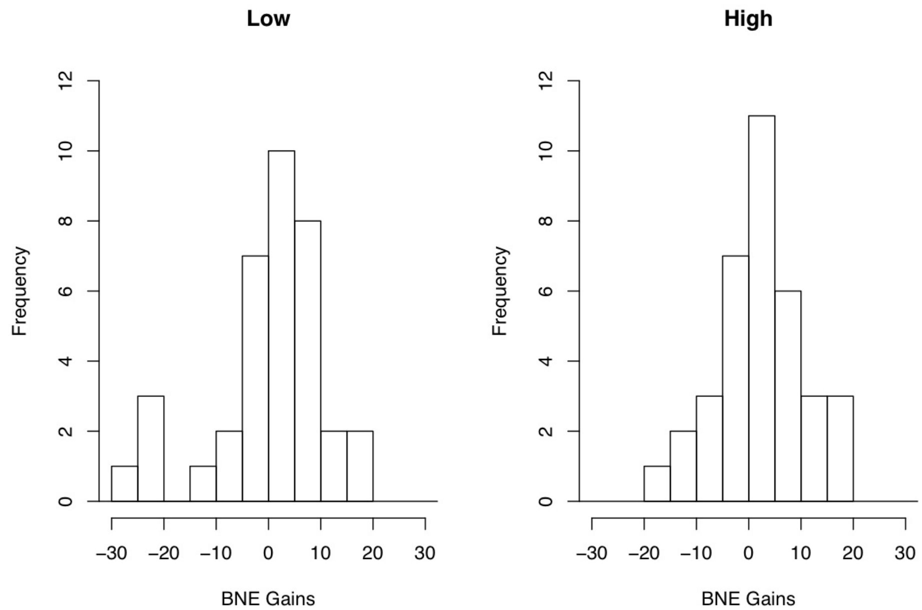

Figure 7: Histograms of gains from unilaterally deviating to BNE strategies calculated by subject.

the earnings obtained from BNE strategies are economically and statistically insignificant.

The finding pertains to individual deviations to BNE. Would earnings significantly increase if all subjects coordinated on BNE strategies? In the Low treatment, such coordination would lead the median subject's earnings to increase by 2.36 points, which is marginally different from zero ($p = 0.0912$). In the High treatment, by contrast, the median subject would earn 4.24 points *less* were they all to coordinate on the BNE ($p = 0.0006$). Thus, BNE play would not substantially improve average earnings in either treatment.

V Discussion

Our exploration of preemption games uncovered several new regularities. On the theoretical side, we were able to extend previous work to obtain precise predictions of behavior in competition. In Bayesian Nash Equilibrium (BNE), each investor waits until the value of the investment opportunity hits a specific threshold that depends on that investor's private cost. Using both a recursion equation and an ordinary differential equation, we characterized the symmetric BNE threshold function for an arbitrary number of competitors and over relevant parameter ranges. Special cases of the parameters give threshold functions identical to known expressions in real options theory and in auction theory. These expressions in turn provide useful bounds and approximations. For example, in triopoly with uniformly distributed costs, the BNE markup of threshold over cost decreases by about \$1 for each \$3 increase in cost.

The laboratory experiment confirmed most of the theoretical predictions. Observed investment thresholds indeed were much lower in triopoly competition than in monopoly. Changes in the parameters driving the stochastic value process had a strong effect (in the predicted direction) in monopoly but (again as predicted) no detectable effect in competition. We also confirmed the sort of efficiency predicted in BNE: lower cost investors are far more likely to preempt than their higher cost rivals.

Other laboratory findings provide more qualified support for the BNE theory. The advantage to following the BNE strategy is much greater at lower costs, and behavior there cannot be distinguished from BNE. At higher costs, however, expected earnings drop quickly and the BNE prediction appears to break down. Thus the data seem consistent with BNE when it matters most, and the data deviate more when deviations make little difference. The upshot is that our subjects on average earn as much as in BNE.

Future laboratory work could investigate behavior for different market structures. We focused on triopoly ($n = 2$) and suspect that higher values of n will push behavior to noisy approximations of the Marshallian threshold. We skipped the duopoly case ($n = 1$) to gain more separation, but future work could investigate it. Perhaps, as in many other laboratory duopolies,[7] there will be frequent attempts to collude. Likewise, future work could investigate different parameter configurations. We chose certain parameter variations a priori, and the results generally conformed to theory. Conceivably the behavioral impact of other parameter variations could differ from the theoretical predictions in interesting ways.

Our experimental design did not support a serious investigation of learning behavior. Behavior generally conformed to equilibrium theory even early on, but the long term trends are unclear. Future research could investigate learning in the preemption game in sessions with 50-100 periods. Such sessions could also manipulate feedback on other players' costs and markup choices in order to identify patterns of social learning. An expanded strategy method might also offer new insights into players' reasoning. After receiving her cost draw, each player could be offered a menu including the option to seize the opportunity manually (as in our experiment) or to program a threshold agent (by filling in the threshold value) or to program any other sort of parametric agent that interests the researcher.

[7] For recent examples, see Martin Dufwenberg and Uri Gneezy (2000) and Steffen Huck, et al (2004)).

Our study examined winner-take-all preemption for several reasons: it brings the central strategic issues into sharp focus, it connects nicely to real options theory and to auction theory, and in many important cases it is a reasonable simplification of reality. (To this list we add that winner-take-all preemption permits clean laboratory implementation via the the semi-strategy method; see Remark A1 in the Appendix.) However, many other important cases involve partial preemption. For example, in larger communities a second or third big box store may be viable, and in some new research areas a second (or even a seventh) paper may be valuable. A clear next step is to extend our work in this direction.

To conclude, our work bridges the gap between two active fields that previously had little contact: real options and auctions. Perhaps our results will lead to more two-way traffic, empirical as well as theoretical.

References

Anderson, Steven T. 2003. "Strategic Timing of Investment: Extensions of Real Options Theory and Timing Games, with Private Information, and an Empirical Application to Foreign Market Entry." Ph.D. Diss. University of California-Santa Cruz.

Anderson, Steven T. , Daniel Friedman, and Oprea, Ryan. "Preemption Games: Theory and Experiment." *University of California, Santa Cruz working paper.*

Barden, Andrew J., and Adriana Arai. 2006. "Bolivia Rules Out Brazil's EBX for $1.1 Bln Project (Update2)." *Bloomberg*, April 24.

Bouis, Romain, Kuno J.M. Huisman, and Peter M. Kort. 2006. "Investment in Oligopoly Under Uncertainty: The Accordion Effect." CentER Discussion Paper 2006-69.

Boyer, Marcel, Éric Gravel, and Pierre Lasserre. 2004. "Real Options and Strategic Competition: A Survey of the Different Modeling Strategies." Paper presented at the Real Options 8th Annual International Conference, Montreal, Canada.

Christensen, Clayton M. 1997. *The Innovator's Dilemma: When New Technologies Cause Great Firms to Fail.* Boston: Harvard Business School Press.

Dixit, Avinash K., and Robert S. Pindyck. 1994. *Investment Under Uncertainty.* Princeton: Princeton University Press.

Dufwenberg, Martin, and Uri Gneezy. 2000. "Price Competition and Market Concentration: An Experimental Study." *International Journal of Industrial Organization*, 18(1): 7–22.

Fudenberg, Drew, and Jean Tirole. 1985. "Preemption and Rent Equalization in the Adoption of New Technology." *Review of Economic Studies*, 52(3): 383–401.

Grenadier, Steven R. 2002. "Option Exercise Games: An Application to the Equilibrium Investment Strategies of Firms." *Review of Financial Studies*, 15(3): 691–721.

Henry, Claude. 1974. "Investment Decisions Under Uncertainty: The Irreversibility Effect." *American Economic Review*, 64(6): 1006–1012.

Hirsch, Morris W., and Stephen Smale. 1974. *Differential Equations, Dynamical Systems, and Linear Algebra.* New York: Academic Press.

Hoppe, Heidrun C., and Ulrich Lehmann-Grube. 2005. "Innovation Timing Games: A General Framework with Applications." *Journal of Economic Theory*, 121(1): 30–50.

Huck, Steffen, Hans-Theo Normann, and Jörg Oechssler. 2004, "Two are Few and Four are Many: Number Effects in Experimental Oligopoly." *Journal of Economic Behavior and Organization*, 53(4): 435–446.

Kaplan, E.L., and Paul Meier. 1958. "Nonparametric estimation from incomplete observations." *Journal of the American Statistical Association*, 53: 457–481.

Karlin, S. and H.M. Taylor 1975. *A First Course in Stochastic Processes.*. NY: Academic Press.

Kreps, David M. 1990. *A Course in Microeconomic Theory.* Princeton: Princeton University Press.

Krishna, Vijay. 2002. *Auction Theory.* Academic Press.

Lambrecht, Bart M., and William R.M. Perraudin. 2003, "Real Options and Preemption Under Incomplete Information." *Journal of Economic Dynamics and Control*, 27(4): 619–643.

Levin, Dan, and James Peck. 2003. "To Grab for the Market or to Bide One's Time: A Dynamic Model of Entry," *RAND Journal of Economics*, 34(3): 536–556.

Levin, Dan, and James Peck. forthcoming. "Investment Dynamics with Common and Private Values." *Journal of Economic Theory.*

McDonald, Robert, and Daniel Siegel. 1986. "The Value of Waiting to Invest." *Quarterly Journal of Economics*, 101(4): 707–728.

McDonnell, Patrick J. 2008. "Bolivia opening up huge iron ore deposit to mining." *Los Angeles Times*, May 28.

Mikesell, Raymond F., and William H. Bartsch, Jack N. Behrman, Phillip E. Church, Gertrud G. Edwards, Henry Gomez, William G. Harris, Markos Mamalakis, Donald A. Wells, Miguel S. Wionczek, James E. Zinser. 1974. "Foreign Investment in the Petroleum and Mineral Industries: Case Studies of Investor-Host Country Relations." For *Resources for the Future, Inc.* Baltimore: Johns Hopkins Press.

Myerson, Roger B. 1981. "Optimal Auction Design." *Mathematics of Operation Research*, 6(1): 58-73.

Oprea, Ryan, Daniel Friedman, and Steven T. Anderson. forthcoming. "Learning to Wait: A Laboratory Investigation." *Review of Economic Studies.*

Reinganum, Jennifer F. 1981. "Dynamic Games of Innovation." *Journal of Economic Theory*, 25(1): 21–41.

Vickrey, William S. 1961. "Counterspeculation, Auctions, and Competitive Sealed Tenders." *Journal of Finance*, 16(1): 8–37.

Rudin, Walter 1966. *Real and Complex Analysis.* NY: McGraw-Hill.

Weeds, Helen. 2002. "Strategic Delay in a Real Options Model of R&D Competition." *Review of Economic Studies*, 69(3): 729–747.

A Mathematical Details

1 Monopoly: Some Known Results

Let the gross value of investment V be governed by the stochastic differential equation

$$dV = \alpha V dt + \sigma V dz, \tag{A1}$$

where z is the standard Wiener process. A key insight from real options theory (e.g., Samuel Karlin and Howard M. Taylor (1975); see also Appendix A of our working paper, Steven T. Anderson et. al. (2008), for a streamlined two-page derivation) is that when future values are discounted at rate $\rho \geq 0$, the expected present value of waiting for appreciation $R > 1$ is $R^{-\beta}$, where

$$\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left[\frac{\alpha}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}} \geq 0. \tag{A2}$$

Thus β is a composite discount parameter that incorporates the stochastic appreciation trend α and volatility σ as well as impatience ρ. In particular, if the current gross value is V_o and the investor will seize the opportunity when V hits threshold $V_1 > V_o$, then the discount factor is

$$\left[\frac{V_o}{V_1}\right]^{\beta}. \tag{A3}$$

When there are no rivals ($n = 0$), an investor with avoidable fixed cost $C > 0$ seeks to maximize the expected discounted profit $E\left[(V-C)\,e^{-\rho t}\right]$. In view of (A3), the problem reduces to finding a threshold V_1 to maximize $[V_1 - C][V_o/V_1]^\beta$. The associated first order condition is

$$0 = \frac{1}{V_1 - C} - \frac{\beta}{V_1}, \tag{A4}$$

with solution

$$V_1 = V_M \equiv \left(\frac{\beta}{\beta - 1}\right) C. \tag{A5}$$

If $\beta > 1$, then (A5) gives the optimal threshold we seek. If $\beta \in [0, 1]$, e.g., if $\alpha \geq \rho$ in (A2), then the expected discounted profit increases in V_1 over the entire domain $V_1 \geq C$, so there is no finite optimal threshold.

Note that (A4-A5) can also usefully be written

$$V_M(C) - \beta[V_M(C) - C] = 0. \tag{A6}$$

2 The Preemption Game: Some Useful Constructs

Let $\Gamma[\beta, n, H]$ be the preemption game described informally in section I.B of the text. A bit more formally, in $\Gamma[\beta, n, H]$ each player ("investor") $i = 1, ..., n+1$ chooses a measurable threshold function $V_i : [C_L, C_U] \to [0, \infty)$, draws realized cost C_i independently from cumulative distribution H on $[C_L, C_U]$, and seizes the investment opportunity at the first time t_i that V in (A1) hits $V_i(C_i)$. In view of (A3), player i's (expected) payoff is

$$[V_i(C_i) - C_i]\left[\frac{V_o}{V_i(C_i)}\right]^\beta \Pr[t_i < t_j \;\; \forall j \neq i], \tag{A7}$$

given initial value V_o in (A1).

We seek a symmetric Bayesian Nash equilibrium (SBNE) of $\Gamma[\beta, n, H]$, that is, a single threshold function V^* such that if all other investors $j \neq i$ choose $V_j = V^*(C_j)$ then investor i maximizes payoff (A7) at any cost draw $C_i \in [C_L, C_U]$ by choosing threshold $V^*(C_i)$. Theorem 1 asserts that such a SBNE V^* exists and is unique, and Theorem 2 establishes some of its general properties. The remainder of this section derives some constructs—in particular, a boundary value problem and a recursion formula—that will help prove those theorems and will provide additional insight.

To begin, suppose all rivals use the same increasing differentiable threshold function $\tilde{V}(C_j)$. At some arbitrary time t_o, let $V_o = V(t_o)$ be the current gross investment value, let $\hat{V} = \max_{s\in[0,t_o]} V(s)$ be the highest value yet observed, and let $\hat{C} = \tilde{V}^{-1}(\hat{V})$ be highest cost draw that would already have led a rival to invest. Investor i can assume that all $C_j > \hat{C}$, since otherwise the game is already over and his choice is moot.

The unconditional probability that any one rival j has cost C_j higher than C is $1 - H(C)$, and the probability conditional on $C_j > \hat{C}$ is $\frac{1-H(C)}{1-H(\hat{C})}$. If investor i chooses threshold $V_1 = \tilde{V}(m)$, the probability that any particular rival will not preempt therefore is $\frac{1-H(m)}{1-H(\hat{C})}$. Thus the conditional probability that none of the n rivals will preempt at that threshold is

$$\Pr[t_i < t_j \quad \forall j \neq i] = \left[\frac{1 - H(m)}{1 - H(\hat{C})}\right]^n \tag{A8}$$

When impatience is due only to expiration hazard, the expression (A3) is simply the probability that the investment opportunity does not expire (or that "Nature does not preempt") before the threshold V_1 is hit. In this case, at current state $[V_o, \hat{V} = \tilde{V}(\hat{C})]$, the probability that investor i "wins" the preemption game is

$$1 - G(m) \equiv \left[\frac{V_o}{\tilde{V}(m)}\right]^\beta \left[\frac{1 - H(m)}{1 - H(\hat{C})}\right]^n. \tag{A9}$$

More generally, (A3) discounts the investor's future values for everything other than preemption by other players, which is captured in (A8), so the product (A9) is the overall discount factor.

Assume that the cost distribution H has density h with full support on the interval $[C_L, C_U]$, where $0 < C_L < C_U < \infty$. The definition (A9) of G then ensures that G has a positive and continuous derivative g. Note also that $1 - G(C_U) = 0$ since $1 - H(C_U) = 0$.

Given investment cost C_i, investor i's problem reduces to finding a threshold $V_1 = \tilde{V}(m)$ that solves

$$\max_{m\in[\hat{C},C_U]} \left[\tilde{V}(m) - C_i\right] [1 - G(m)]. \tag{A10}$$

The FOC is

$$\tilde{V}'(m)[1 - G(m)] - \left[\tilde{V}(m) - C_i\right] g(m) = 0. \tag{A11}$$

In SBNE, investor i will find it advantageous to use the same threshold function $\tilde{V}$ as the other investors. Accordingly, insert the "truthtelling" condition $m = C_i$ into (A11), and simplify notation by setting $C_i = C$, to obtain

$$\tilde{V}'(C)[1 - G(C)] - \left[\tilde{V}(C) - C\right] g(C) = 0. \tag{A12}$$

Rearranging slightly, write

$$-Cg(C) = \tilde{V}'(C)[1 - G(C)] - \tilde{V}(C)g(C) = \frac{d}{dC}[\tilde{V}(C)(1 - G(C))], \tag{A13}$$

and integrate both sides of (A13) from $C = C_i$ to C_U, using $1 - G(C_U) = 0$, to obtain

$$-\int_C^{C_U} yg(y)dy = 0 - \tilde{V}(C)[1 - G(C)]. \tag{A14}$$

Hence the SBNE threshold function $\tilde{V} = V^*$ must satisfy

$$V^*(C) = \frac{1}{1 - G(C)} \int_C^{C_U} yg(y)dy \equiv E_G[y|y > C]. \tag{A15}$$

Since H and hence G are smooth strictly increasing functions, equation (A15) ensures that V^* is also smooth and increasing in C.

Of course, the function G is itself defined in terms of V^*.[8] To obtain an explicit recursion formula, first integrate by parts to get $\int_C^{C_U} yg(y)dy = C_U - CG(C) - \int_C^{C_U} G(y)dy$. Then (A15) reads

$$\begin{aligned} V^*(C) &= \frac{1}{1 - G(C)} \left[C_U - CG(C) - \int_C^{C_U} 1 - \left(\frac{V_o}{V^*(y)}\right)^\beta \left(\frac{1 - H(y)}{1 - H(\hat{C})}\right)^n dy \right] \\ &= \frac{1}{1 - G(C)} \left[C[1 - G(C)] + \int_C^{C_U} \left(\frac{V_o}{V^*(y)}\right)^\beta \left(\frac{1 - H(y)}{1 - H(\hat{C})}\right)^n dy \right] \\ &= C + \int_C^{C_U} \left[\frac{V^*(C)}{V^*(y)}\right]^\beta \left[\frac{1 - H(y)}{1 - H(C)}\right]^n dy. \end{aligned} \tag{A16}$$

The last expression shows that $V^*(C)$ is equal to C plus a positive markup, which shrinks to 0 as C approaches its upper endpoint $C = C_U$.

Remark A1. Note that V_o and $\hat{V} = V^*(\hat{C})$ drop out of the formula (A16) for V^*. This reflects the fact that the opportunity to observe rivals' actions does not influence thresholds; $V^*(C)$ can be set as soon as the investor draws cost C, and subsequent observations are irrelevant. The logic parallels the strategic isomorphism of Dutch and first price auctions.

[8]Here we depart from the auction literature, e.g., Krishna (2002, pp. 14-19).

To obtain the boundary value problem satisfied by V^*, first insert (A9) into (A10) to write the objective function as in the text:

$$F\left(m|C_i,n\right) = \left[V^*\left(m\right) - C_i\right]\left[\frac{V}{V^*\left(m\right)}\right]^{\beta}\left[\frac{1-H\left(m\right)}{1-H\left(\hat{C}\right)}\right]^{n}. \tag{A17}$$

Use the product rule to take the derivative of Equation (A17) with respect to m and cancel like terms (or, alternatively, take the derivative of $\ln F$) and evaluate at the "truth-telling" point $m = C_i$, to obtain the FOC:

$$\frac{V^{*\prime}\left(C_i\right)}{\left[V^*\left(C_i\right) - C_i\right]} - \frac{\beta V^{*\prime}\left(C_i\right)}{V^*\left(C_i\right)} - \frac{nh\left(C_i\right)}{\left[1-H\left(C_i\right)\right]} = 0. \tag{A18}$$

Solve (A18) for $V^{*\prime}$ to obtain Equation (5), reproduced here for convenience:

$$V^{*\prime}(C_i) = \frac{\left[V^*\left(C_i\right) - C_i\right]V^*\left(C_i\right)}{V^*\left(C_i\right) - \beta\left[V^*\left(C_i\right) - C_i\right]} \times \frac{nh(C_i)}{\left[1-H\left(C_i\right)\right]} \tag{A19}$$

As noted in the text, the boundary condition

$$V^*(C_U) = C_U \tag{A20}$$

comes from the economics of the situation. At the highest possible cost realization, the existence of rivals known to have equal (or lower) cost induces Bertrand competition and drives the markup to zero.

Remark A2. To obtain V^* numerically, one can use the Euler method of integrating the ODE (A19) backward from the upper boundary value (A20). Alternatively, one can take an initial approximation such as the auction solution $\bar{V}$, substitute it for V^* in the last expression in (A16) to obtain a better approximation, and iterate. The BNE threshold function V^* is a fixed point of this mapping.

3 Proof of Theorem 1

Theorem 1. *Let the cumulative distribution function H have a continuous density h with full support $[C_L, C_U]$, where $0 < C_L < C_U < \infty$. Let it be common knowledge among all investors $i = 1, ..., n+1$ that i's investment cost C_i is an independent random variable with distribution H that is observed only by investor i. Then*

1. *for any $\beta \geq 0$, boundary value problem (A19-A20) has a unique solution $V^* : [C_L, C_U] \to R$,*

2. *a function V^* satisfies the recursion equation (A16) iff it solves the boundary problem (A19-A20), and*

3. *the premption game $\Gamma[\beta, n, H]$ has a symmetric Bayesian-Nash equilibrium in which each investor i's threshold is V^* evaluated at realized cost C_i.*

Lemma A1. *Assume that $\beta > 1$, that all rivals use a threshold function with inverse γ such that $\gamma' > 0$, and that the hypotheses of Theorem 1 hold. Let the threshold value y maximize the competitor's payoff given cost realization $C \in [C_L, C_U]$. Then $y < V_M^*(C)$.*

Proof of Lemma. Write (A4) as

$$0 = \frac{1}{x - C} - \frac{\beta}{x}. \tag{A21}$$

and write (A17) as

$$[x - C]\left[\frac{V}{x}\right]^{\beta}\left[\frac{1 - H(\gamma(x))}{1 - H(\hat{C})}\right]^{n}. \tag{A22}$$

Since the threshold value $x = y$ maximizes (A22), it must satisfy the FOC

$$0 = \frac{1}{x - C} - \frac{\beta}{x} - \frac{nh[\gamma(x)]\gamma'(x)}{1 - H[\gamma(x)]}. \tag{A23}$$

By (A21), at $x = V_M^*$ the RHS of (A23) reduces to

$$-\frac{nh[\gamma(x)]\gamma'(x)}{1 - H[\gamma(x)]} < 0. \tag{A24}$$

Since the RHS of (A21) is negative for $x > V_M$, no value of $x \geq V_M$ can satisfy the FOC (A23). On the other hand, since the first RHS term in (A23) goes to $+\infty$ as $x \searrow C$ while the other terms remain bounded, the continuity of the RHS in x guarantees a solution to (A23) at some value $x = y \in (C, V_M)$. $\diamondsuit$

Proof of Theorem 1. The key step in part 1 is to show that the RHS of the ODE (A19) is Lipschitz continuous in V^*. A first potential problem is that the denominator factor $1 - H(C) \searrow 0$ as $C \nearrow C_U$. But given the boundary condition (A20), the numerator factor $V^*(C) - C \searrow 0$ also.

Indeed, we have

$$
\begin{aligned}
V^{*\prime}(C_U) &= \lim_{C \nearrow C_U} \frac{[V^*(C) - C]V^*(C)}{V^*(C) - \beta[V^*(C) - C]} \times \frac{nh(C)}{1 - H(C)} \\
&= \frac{nh(C_U)V^*(C_U)}{V^*(C_U) - \beta[V^*(C_U) - C_U]} \lim_{C \nearrow C_U} \frac{V^*(C) - C}{1 - H(C)} \\
&= \frac{nh(C_U)}{1 - \beta[1 - \frac{C_U}{C_U}]} \frac{V^{*\prime}(C_U) - 1}{[-h(C_U)]} = -n[V^{*\prime}(C_U) - 1], \qquad \text{(A25)}
\end{aligned}
$$

using L'Hospital's rule. Hence $V^{*\prime}(C_U) = n/(n+1)$, (independent of β and H!). In particular, we have Lipschitz continuity in an ϵ neighborhood of the upper boundary point.

A second potential problem is that the other denominator factor $V^*(C) - \beta[V^*(C) - C]$ might be zero (or negative). For $\beta \leq 1$ the factor obviously is positive. Lemma A1 assures us that the expression is also positive when $\beta > 1$:

$$
\begin{aligned}
V^*(C) - \beta[V^*(C) - C] &= y - \beta[y - C] \\
= \beta C - (\beta - 1)y &> \beta C - (\beta - 1)V_M(C) \\
&= V_M(C) - \beta[V_M(C) - C] = 0, \qquad \text{(A26)}
\end{aligned}
$$

since $V^*(C) = y < V_M(C)$ by the Lemma and the last expression is 0 by (A6).

Since the denominator is continuous on the closed interval $[C_L, C_U - \epsilon]$, it achieves a positive minimum value and thus is bounded away from zero. It is now clear that the RHS of the ODE (A19) is positive, bounded and Lipschitz continuous in V^*. The classic Picard-Lindelof theorem (e.g., see Chapter 8 of Morris W. Hirsch and Stephen Smale, 1974) then guarantees that a solution V^* to the boundary problem (A19-A20) exists and is unique.

For the second part of the Theorem, suppose that V^* satisfies the recursion equation (A16). Note that the integrand $f(C, y)$ in the last term of (A16) is the product of two increasing functions of C, so $\frac{\partial f(C,y)}{\partial C} > 0$. Indeed, using the product rule, one obtains $\frac{\partial f(C,y)}{\partial C} = \left(\frac{\beta V'(C)}{V(C)} + \frac{nh(C)}{1-H(C)}\right) f(C, y)$. Differentiating both sides of (A16), we get

$$
\begin{aligned}
V^{*\prime}(C) &= 1 - f(C, C) + \int_C^{C_U} \frac{\partial f(C, y)}{\partial C} dy = \int_C^{C_U} \frac{\partial f(C, y)}{\partial C} dy \\
&= \left(\frac{\beta V^{*\prime}(C)}{V^*(C)} + \frac{nh(C)}{1 - H(C)}\right) \int_C^{C_U} f(C, y) dy \\
&= \left(\frac{\beta V^{*\prime}(C)}{V^*(C)} + \frac{nh(C)}{1 - H(C)}\right) [V^*(C) - C] > 0 \qquad \text{(A27)}
\end{aligned}
$$

for all $C \in [C_L, C_U)$. The last equality follows from subtracting C from both sides of (A16). Divide (A27) through by the markup $V^*(C) - C > 0$ to obtain (A18). As noted earlier, this can be rewritten as the ODE (A19). In the limit as $C \nearrow C_U$, the integral term vanishes in (A16) and we obtain the boundary condition (A19).

Conversely, suppose that V^* solves the boundary value problem (A19-A20). Then the derivation (A12-A16) ensures that it also satisfies the recursion formula (A16).

To establish part 3 of the Theorem, let $W(C, m)$ be the expected payoff to investor i when she draws cost C and and employs threshold $V_1 = V^*(m)$, assuming each other investor j sets threshold $V^*(C_j)$. We complete the proof by showing that setting $m = C$, i.e., truthtelling, always maximizes $W(C, m)$.

Assume that $m > \hat{C}$; otherwise the investor already would have ended the game. Consequently $V^*(m) > \hat{V} \geq V_o$ and, by (A10) and (A15),

$$\begin{aligned} W(C, m) &= [V^*(m) - C][1 - G(m)] \\ &= \int_m^{C_U} y g(y) dy - C + CG(m) \\ &= [C_U - C] + [C - m]G(m) - \int_m^{C_U} G(y) dy, \end{aligned} \tag{A28}$$

where the last expression uses integration by parts: $\int_m^{C_U} y g(y) dy = C_U - mG(m) - \int_m^{C_U} G(y) dy$.

But (A28) and the Mean Value Theorem (MVT) yield a point x such that

$$\begin{aligned} W(C, C) - W(C, m) &= [m - C]G(m) + \int_m^C G(y) dy \\ &= [m - C]G(m) + [C - m]G(x) \\ &= [m - C][G(m) - G(x)] \geq 0, \end{aligned} \tag{A29}$$

since G is increasing and the number x guaranteed by the MVT is between m and C. Hence $m = C$ is indeed a best response and V^* is indeed a symmetric BNE. ♢

Remark A3. Theorem 1 imposes the hypothesis that H have a continuous positive density on the entire support interval, and that the upper endpoint is finite and the lower endpoint is positive. It follows from Lusins theorem (e.g., Walter Rudin, 1966, pp.53-54) that such functions are a dense subset of all distribution functions on $[0, \infty)$, so this restriction is not especially onerous. Still, it would be of interest to explicitly consider distributions with discrete support (e.g., a finite set of cost "types") or asymmetric cases in which some investors are known to have different cost distributions than other investors. Perhaps "ironing" techniques in the spirit of Roger B. Myerson (1981) would be useful for such extensions.

4 Proof of Theorem 2

Proof of Theorem 2. The RHS of ODE (A19) is positive and continuous, so its solution V^* is increasing and continuously differentiable. The Marshallian lower bound is also obvious: a threshold below cost implies negative profit, but zero profit can be assured in this game by never investing. Hence any BNE threshold must be at least the realized cost.

To see that V_M is an upper bound for the SBNE threshold, note that the investor seeks threshold V_1 to maximize

$$[V_1 - C]\left[\frac{V_o}{V_1}\right]^{\beta}\left[\frac{1-H(V^{*-1}(V_1))}{1-H(\hat{C})}\right]^{n}, \tag{A30}$$

The optimum threshold value $V_1 = V^*(C)$ must satisfy the FOC

$$0 = \frac{1}{V_1 - C} - \frac{\beta}{V_1} - \frac{nh[V^{*-1}(V_1)]V^{*-1\prime}(V_1)}{1-H[V^{*-1}(V_1)]}. \tag{A31}$$

By (A4), at $V_1 = V_M$ the RHS of (A31) reduces to

$$-\frac{nh[V^{*-1}(V_1)]V^{*-1\prime}(V_1)}{1-H[V^{*-1}(V_1)]} < 0. \tag{A32}$$

Since the RHS of (A4) is negative for $V_1 > V_M(C)$, no value of $V_1 \geq V_M$ can satisfy the FOC (A31). On the other hand, since the first RHS term in (A31) goes to $+\infty$ as $V_1 \searrow C$ while the other terms remain bounded, the continuity of the RHS in V_1 guarantees a solution to (A31) at some value $V_1 \in (C, V_M(C))$, i.e., V_M is indeed an upper bound.

A similar argument shows that the auction bid function $\bar{V}$ is also an upper bound. Evaluating the RHS of (A31) at $V_1 = \bar{V}(C)$, the first and third terms disappear due to the FOC that characterizes $\bar{V}$, and the remaining term, $-\beta/V_1$, is negative. Again, higher values of V_1 only make the RHS of (A31) more negative, but we know that the RHS is positive for sufficiently small values of V_1. Hence the intermediate value theorem again guarantees a solution to (A31) at some value $V_1 \in (C, \bar{V}(C))$.

Recall that (A25) showed that $V^{*\prime}(C_U) = n/(n+1)$, independent of β. Since $V^* = \bar{V}$ in the special case $\beta = 0$, we have $\bar{V}'(C_U) = n/(n+1) = V^{*\prime}(C_U)$. Of course, $V^*(C_U) = C_U = \bar{V}(C_U)$, so $\bar{V}$ and V^* are indeed tangent at C_U. $\diamondsuit$

Remark A4. It is well known from auction theory that the bid function $\bar{V}(C)$ converges to the Marshallian threshold function $V^0(C) = C$ as the number of rivals $n \to \infty$. A simple consequence of Theorem 2 is that the same is true of the BNE threshold function $V^*(C)$. As n gets large, its upper bound $\bar{V}$ converges to its lower bound V^0, so in the large number limit, $V^*(C) = V^0(C) = C$. That is, markups converge to zero as the number of rivals increases.

5 Other results

The text following the statement of Theorem 2 listed conditions guaranteeing that the threshold function is concave. We now formalize and prove that result (the only non-trivial part of the Corollary).

Proposition A1. *Suppose that the hypotheses of Theorem 1 hold and that the density h of the cumulative distribution function H is non-increasing. Then the BNE threshold function $V^*(C)$ is concave.*

Proof. A sufficient condition for concavity is that $V^{*\prime\prime}(C) \leq 0$, so it sufficies to show that the derivative with respect to C of the RHS of (A19) is not positive. Direct computation reveals that that derivative is

$$\frac{N(C|H,\beta)}{D} = \frac{\{[V^* - \beta\,(V^* - C)]\,[1-H]\} \times \{(nh)\,[(2V^* - C)\,V^{*\prime} - V^*] + [(nh')\,(V^* - C)\,V^*]\}}{\{V^* - \beta\,[V^* - C] \times [1-H]\}^2} - \frac{\{[(1-\beta)V^{*\prime} + \beta]\,[1-H] - (h)\,[V^* - \beta(V^* - C)]\} \times \{(nh)\,(V^* - C)\,V^*\}}{\{V^* - \beta\,[V^* - C] \times [1-H]\}^2}. \quad \text{(A33)}$$

The proof of Theorem 1 shows that $V^* - \beta[V^* - C]$ is bounded below by a positive number, so clearly the denominator D is positive. Thus it suffices to show that the numerator $N \leq 0$.

Now note that h' appears only once in N and that the other factors in that term are positive. Hence if the proposition holds for a uniform distribution, then it holds *a forteriori* for one with decreasing density. The rest of this proof therefore assumes that the distribution is uniform.

Write

$$N(C|H,\beta) = N(C|H,0) + \{[-\beta\,(V^* - C)]\,[1-H]\} \times (nh)\left[(2V^* - C)\,V^{*\prime} - V^*\right] - \{\left[\beta - \beta V^{*\prime}\right][1-H] + (h)\beta(V^* - C)\} \times \{(nh)\,(V^* - C)\,V^*\} \quad \text{(A34)}$$

We know that the (Vickrey) bid function is linear and hence $N(C|H,0) = 0$ when H is uniform.

Factoring out $nh\beta(V^* - C)$ from (A34), and simplifying, we obtain

$$N(C|H,\beta) = -nh\beta(V^* - C)^2 \left[(1-H)V^{*\prime} + hV^*\right], \tag{A35}$$

which is strictly negative when $\beta > 0$. $\diamondsuit$

Remark A5. The main results from Anderson (2003) are subsumed in this Appendix, with one exception. He obtains a comparative static result for mean-preserving spreads of the cost distribution H.

6 Brownian Parameters

Dixit and Pindyck (1994, pp 69-70) show that the deviation of the uptick probability p from 0.5, times the distance 2η between an uptick and downtick, corresponds to the Brownian drift rate α:

$$\alpha = \lim_{\Delta t \to 0} \frac{(2p-1)\,\eta}{\Delta t}. \tag{A36}$$

The Brownian volatility σ comes mainly from the stepsize η but when p differs from 0.5 we must also account for binomial variance $p\,(1-p)$. The exact expression is

$$\sigma^2 = \lim_{\Delta t \to 0} \frac{4p\,(1-p)\,\eta^2}{\Delta t}. \tag{A37}$$

The relation between expiration probability $q \in [0,1]$ and the discount parameter $\rho > 0$ is obtained as follows. By definition of discount, 1 unit value received a time unit from now is equivalent to $e^{-\rho}$ units received immediately. The usual interpretation of such impatience is the foregone interest payments available in a financial market. For practical reasons, we turn to an alternative interpretation (e.g., David Kreps, 1990, p.505-6): the discount rate arises from the possibility that the opportunity will expire. If that event has probability Q per unit time, then the expected value of 1 unit due a time unit in the future is $1 - Q$. With $T = 1/\Delta t$ time steps per unit time, and expiration probability q per time step, the discount factor is $e^{-\rho} = 1 - Q = (1-q)^T = (1-q)^{1/\Delta t}$. Solving for ρ we obtain

$$\rho = \frac{-\ln(1-q)}{\Delta t}. \tag{A38}$$

Note that Q is expressed per unit time, while q corresponds to a fixed Δt. Hence in (A38) we don't take the limit $\Delta t \to 0$.