The S-Shaped Value Function as a Constrained Optimum

By DANIEL FRIEDMAN*

Almost since the Expected Utility Hypothesis (EUH) was first introduced, evidence has accumulated that decision makers systematically violate it in various ways. One early response to the evidence was to develop *ad hoc* alternatives, the best known of which is Prospect Theory (Daniel Kahneman and Amos Tversky, 1979).[1] More recent responses have relaxed the independence axiom or other axioms underlying the EUH. See Mark Machina (1987) for a very readable survey of the evidence and responses. Despite the high intellectual caliber of much of this work, there is an important sense in which it has been retrograde: theory is adjusted to the evidence by weakening, not strengthening, its predictive power.

In other areas (such as the theory of the firm or the theory of money) economists have followed a different research strategy: more subtle constraints (such as transactions costs or informational imperfections) are sought to explain data that seem anomalous according to received theory. When successful, such a strategy more clearly delineates the realm in which the older theory is applicable and makes new, testable predictions outside that realm. Thus theory is progressively strengthened.

To my knowledge, only Jonathan Leland (1986, 1988) has adopted such a progressive strategy in a theoretical investigation of EUH anomalies. The key assumptions of his approximate expected utility theory (AEU) are (1) inexperience and/or cognitive limitations make the "true" utility function inaccessible, so some approximation must be used; but (2) decision makers efficiently allocate finite resources so as to minimize the resulting errors. He formalizes these assumptions (which he applies to probability assessments as well as to utilities) in terms of a constraint on the number N of steps allowed in a step-function approximation, with the optimal approximation defined as that minimizing expected squared error. His AEU theory is able to explain many of the major EUH anomalies, and reduces to ordinary EUH in areas in which relevant experience accumulates (for example, in competitive markets; as noted by Peter Knez, Vernon L. Smith, and Arlington W. Williams (1985); and Don L. Coursey, John L. Hovis, and William D. Schultze (1987), the prevalence and magnitude of anomalies seems to decline in such settings).

The purpose of this note is to present a variant on Leland's AEU. I propose an approximate utility function (or "value function") in which utility increments are weighted by a sensitivity function. The resource constraint is that overall sensitivity is limited, but (prior to observable decisions) it can be allocated freely along the continuum of potential wealth increments.

My approach has three advantages: it *predicts* an S-shaped value function when agents maximize expected sensitivity at actual choice opportunities, given very plausible assumptions regarding the distribution of such opportunities. Recall that the *assumption* that decision makers maximize an S-shaped value function, such as that depicted in Figure 1, is the centerpiece of Prospect Theory,

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[1]The authors of Prospect Theory call their model "descriptive" as opposed to "normative," but do not consider it *ad hoc* because of its connections with the psychophysical literature. However, no substantial connections of Prospect Theory with generally accepted economic principles (for example, optimality) have previously been demonstrated. It is in this narrow economist's sense that I use the term *ad hoc*. The more recent work of Ariel Rubinstein (1988) and Tversky and Kahneman (1986) underscores the difficulty of reconciling such a descriptive approach to the expected utility hypothesis.

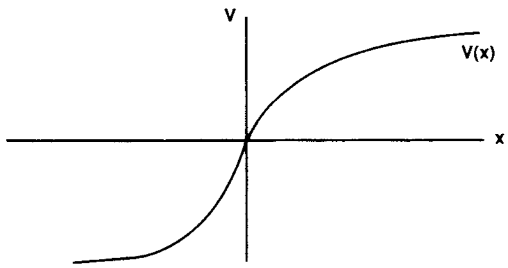

FIGURE 1. AN S-SHAPED VALUE FUNCTION

and explains many EUH anomalies.[2] Second, my approach avoids the counterfactual discontinuities in the value function that arise from a literal interpretation of Leland (1988). Finally, my derivation is very simple and direct. Since my primary intention is to illustrate a progressive approach to EUH anomalies rather than to provide a definitive model, this simplicity is vital, but it does have a cost: my model explains only a subset of the anomalies explained by Leland (1988).[3]

After sketching the formal model in the next section, I derive three results which (1) characterize optimal allocation of sensitivity, (2) give sufficient conditions for the approximate utility ("value") function to be S-shaped, and (3) verify its convergence to the "true" utility function as the resource constraint is relaxed. The final section discusses implications of the model.

I. A Simple Model

I begin with some basic definitions. A function f: $R^1 \to R^1$ is called *smooth* if it is at least twice continuously differentiable. It is *unimodal at* $x_o \in R^1$ if $f'(x) \lesseqgtr 0$ as $x \gtrless x_o$. If also $f''(x_o) < 0$ then f is *strongly*[4] unimodal at x_o. It is *S-shaped*[5] *around* x_o if $f''(x) \lesseqgtr 0$ as $x \gtrless x_o$. For example, a Normal density function is strongly unimodal at its mean μ, and its cumulative distribution function is S-shaped around μ. A function is *locally* unimodal (or S-shaped, etc.) if its restriction to an open neighborhood of x_o is unimodal (or S-shaped, etc.).

A smooth function U is a *utility function* if $U'(x) > 0$ for all x in its domain. Another function V defined on the same domain is *vNM-equivalent* to U if there are constants $a > 0$ and b such that $V(x) = aU(x) + b$ for all x in their common domain. The *absolute risk aversion* (*ara*) *function* for U is $r(x) = -U''(x)/U'(x)$. It is easily seen that two smooth utility functions are vNM-equivalent iff they have the same *ara* function, and iff $V'(x) = aU'(x)$ for some $a > 0$. Such functions are said to *represent the same preferences*.

The substantive assumptions of the model are expressed in terms of a nonnegative *sensitivity function* s: $R^1 \to R^1_+$, assumed square-integrable. Its L_2-norm, $S = |s|_2 = \{\int_{-\infty}^{\infty} s^2(x)dx\}^{1/2}$, is called its *overall sensitivity*. The sensitivity function is related to approximate utility in terms of a *scaling function* h: $R_+ \to [0,1]$ which is smooth and strictly increasing, with $h(0) = 0$ and $h(\infty) = 1$. An example is $h(t) = t(1+t)^{-1}$. Call V an *s-approximation* to U if, for some given scaling function h, we have $V'(x) = ah(s(x))U'(x)$ for all x in the domain of U and some $a > 0$. The idea is that V gives only partial weight to "true" utility increments $\Delta U(x)$, the weight increasing to a full weight of 1 as the sensitivity allocated at x increases.

Let w_o denote an agent's current wealth level, and suppose the ultimate objects of choice consists of lotteries with wealth-increments $x \in [-w_o, \infty)$. Suppose relevant

[2] Peter C. Fishburn and Gary A. Kochenberger (1979), among others, used individual response to estimate utility functions. The predominant result was a convex-concave, that is, S-shaped function.

[3] The model presented here does not seem to predict the common-ratio effect or the *ntg/ntl* effect that Leland (1988) analyzes in terms of imperfect probability assessments, or "decision weighting functions." Leland treats decision weighting functions in a manner analogous to value functions, and perhaps the same could be done under the present approach.

[4] Intuitively, unimodal means single-peaked, and strongly unimodal means the peak is not a plateau, even infinitesimally.

[5] Thus, S-shaped means convex below an inflection point x_o and concave above, like the letter S rotated clockwise slightly.

choice opportunities are not yet known, but their *distribution* is described by a known cumulative distribution function F supported on $[-w_o, \infty)$, whose density function (if it exists) is denoted by f. Suppose also that the agent has latent (or "true") risk preferences represented by a utility function $U(x)$ with a smooth *ara* function $r(x)$ defined on $[-w_o, \infty)$.

The substantive assumptions are:

ASSUMPTION A1: *For some given scaling function h and some chosen sensitivity function s, an agent uses V, an s-approximation to U, in evaluating the ultimate lotteries. That is, observed behavior is described by expected utility maximization with respect to V rather than with respect to U.*

ASSUMPTION A2: *The sensitivity function s is chosen so as to maximize expected sensitivity at realized opportunities* $x \in [-w_o, \infty)$, *subject to the constraint that overall sensitivity is bounded by some finite positive number S.*

The basic idea behind the assumptions is that resources of time, attention, and cognition can be allocated in a pre-decision phase to imagining the marginal benefit of various wealth-increments. (For example, "if I had an extra \$100, how would I spend it and how happy would that make me if I were \$$x$ wealthier than now?," $x = -\$1000, \$0, +\$1000$, etc.). However, these "sensitivity" resources are finite, and it seems reasonable to summarize them by the L_2-norm, which is the continuous analogue of Euclidean distance from the origin.[6] An alternative interpretation (perhaps more appealing to those who do not share most economists' reflexive belief in constrained rationality) is to regard the allocation of sensitivity as the passive result of accumulated experience, direct or indirect. Well-known arguments suggest that approximate optimality can arise from naive learning processes under the pressure of natural selection (for example, Armen Alchian, 1950). Of course, the construction of V from s and U in any case is definitely an "as-if" exercise; if agents could literally do this, they would be able to employ U directly. Agents use V rather than U precisely because they are more sensitive to $\Delta U(x)$ for some values of x (those that they have experienced or thought more about) than others. Under either interpretation it is reasonable to presume that the level S of overall sensitivity increases as relevant experience accumulates.

If the precise nature of the ultimate decision problem were known (for example, binary choice of lotteries, or an insurance decision, or portfolio choice) then (A2) would be replaced by the optimization problem obtained directly from that decision. However, agents typically have many different problems to solve, not all of which can be specified in advance. Hence expected sensitivity maximization is a reasonable proxy for the composite problem. Other variants of Leland's AEU approach might pose the optimization problem differently.

Special attention is given below to distributions of choice opportunities whose density is unimodal at 0. It is perhaps worth noting at the outset that such a distribution is implied by the not unreasonable assumptions that (a) the distribution does have a density and (b) opportunities for smaller wealth increments (gains or losses) arise more frequently than opportunities for larger wealth increments.

II. Results

I first characterize the optimal allocation of given overall sensitivity S, where optimality is defined by Assumption (A2).

PROPOSITION 1: *If the distribution of potential wealth increments has a density* $f(x)$ *which is square-integrable on* $(-\infty, \infty)$, *then the optimal sensitivity function s is proportional to f and to S, that is,* $s(x) = cf(x)$ *for almost every* $x \in (-\infty, \infty)$ *and some positive constant c which itself is proportional to S.*

[6] One could also consider the problem of allocating sensitivity to a given discrete set of outcomes $x_1, \ldots, x_n$, using the root mean squared sum of sensitivities to summarize the overall level of sensitivity. The L_2-norm is again the continuous analogue.

PROOF:

From (A2) and the distributional hypothesis, the optimization problem is

$$(1) \quad \max E_f s = \int_{-\infty}^{\infty} s(x)f(x)dx \quad \text{s.t.} \int_{-\infty}^{\infty} s(x)^2 dx = S^2.$$

Applying the Cauchy-Schwarz inequality (see Royden, 1968, p. 210, for instance), we find that $E_f s \leq |f|_2 \cdot S$, where $|f|_2^2 = \int_{-\infty}^{\infty} f(x)^2 dx$, and that equality holds iff $s(x) = cf(x)$ a.e., where $c = S/|f|_2$.[7] □

The next result shows that the optimally calibrated value function V will be S-shaped when f is unimodal, at least if the *ara* function is not too large. Indeed, in the risk-neutral case $r(x) = 0$, V will inherit its shape directly from the distribution function F, which of course is S-shaped when f is unimodal.

PROPOSITION 2: *Let V be an s-approximation of the true utility function U, whose absolute risk aversion function is $r(x)$. If s is optimal with respect to a distribution of wealth increments with smooth density f, then (a) V is smooth and S-shaped around 0 if f is unimodal at 0 and $r(x) = 0$, and (b) V is smooth and locally S-shaped around a negative (resp. positive) number x_o if f is strongly unimodal at 0, and if $|r(0)|$ and $|r'(0)|$ are sufficiently small with $r(0)$ positive (resp. negative).*

PROOF:

Since V is an s-approximation of U, there is some constant $a > 0$ and some scaling function h: $[0,\infty) \to [0,1)$ so that $V'(x) = ah(s(x))U'(x)$. Hence V is smooth and from Proposition 1 and the definition of *ara* functions we obtain

$$\begin{aligned} V''(x) &= ah(s(x))U''(x) + ah'(s(x))s'(x)U'(x) \\ &= b(x)\{s'(x)h'(s(x))/h(s(x)) - r(x)\} \\ &= b(x)\{g(x)f'(x) - r(x)\}, \end{aligned}$$

[7] The result can also be obtained under stronger assumptions on f from the Euler equations for (1).

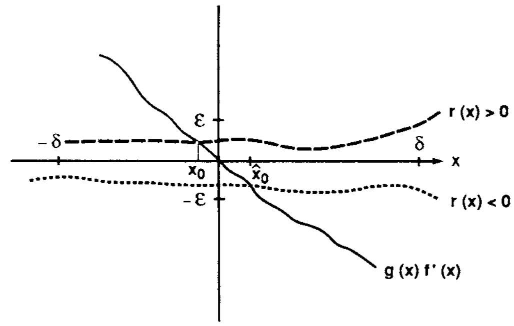

FIGURE 2. LOCATION OF INFLECTION POINT x_0

where the functions $b(x) = ah(s(x))U'(x)$ and $g(x) = S|f|_2^{-1}h'(s(x))/h(s(x))$ are smooth and positive for $x \in (\text{supp} f)^o$. Thus the sign of V'' is the sign of the term in braces. In the $r(x) = 0$ (risk-neutral) case, the latter is therefore the sign of f', which (since f is unimodal at 0) is the sign of $-x$. Hence by definition V is (globally) S-shaped around 0 in this case.

As for part (b), the extra hypotheses imply that there is some interval $I = (-\delta, \delta)$ around 0 and some $\varepsilon > 0$ so that the term in braces has exactly one root x_o in I if $|r(0)| + |r'(0)| < \varepsilon$. It is easy to check that x_o has sign opposite to that of $r(0)$, as shown in Figure 2. The same argument as in part (a) then shows that V restricted to I is S-shaped around x_o. □

Finally, I confirm that as experiences or cognitive resources accumulate, as indicated by an increasing overall sensitivity S, then agents will use closer approximations of the true utility function U. See Figure 3 for an illustration.

PROPOSITION 3: *Let V^S be an optimal s-approximation to U for overall sensitivity $S > 0$. If the distribution F of potential wealth increments has a density f, then (a) $\lim_{S \to 0} V^S$ is vNM-equivalent to a constant function, and (b) $\lim_{S \to \infty} V^S$ is vNM-equivalent to U.*

PROOF:

Since h is continuous, $h(s(x)) = h(cf(x)) = h(Sf(x)/|f|_2)$ converges to $h(0) = 0$ as

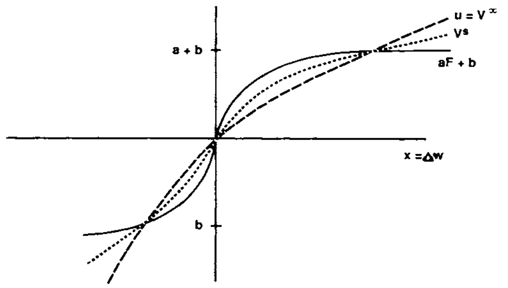

FIGURE 3. CONVERGENCE OF V

$S \to 0$ and to $h(\infty)=1$ as $S \to \infty$ for each $x \in [-w_0, \infty)$. Hence $V(x)$ converges pointwise to 0 in case (a) and to $aU(x)+b$ in case (b). □

III. Implications

The main insight arising from the model just presented is that an individual will typically not make the same choices in actual decision problems as he would make if he had unlimited experience and/or the time and inclination for unlimited introspection. Instead, given "sensitivity" limitations, he will choose as if maximizing expected value for a value function V that in some sense is between his "true" fully considered preference function U and the cumulative distribution function F of prospective opportunities to increment wealth.

Specific predictions of the model therefore depend largely on the distribution of prospective opportunities. Most individuals face opportunities for smaller gains or losses more frequently than opportunities for larger gains or losses, so it is reasonable to suppose that the density function f is unimodal at 0 and hence the distribution function F is S-shaped around 0. It then follows that the value function V will also tend to be S-shaped near current wealth. Furthermore, noting that the upper bound on wealth increments is world-aggregate wealth while the lower bound is $-$personal wealth, one could reasonably argue that f is positively skewed; a straightforward derivation (omitted from the previous section for the sake of brevity) then shows that V is typically steeper for losses than for gains.

Kahneman and Tversky (1979) showed that such a value function can account for much behavior that seems anomalous according to the Expected Utility Hypothesis. First (as in Milton Friedman and Leonard Savage, 1948) it implies risk seeking at some wealth levels and risk aversion at higher levels. However, unlike the Friedman-Savage utility function, it predicts that individuals at all income levels have an inflection point at (or very near) current wealth, which is much more consistent with observed behavior. The skewness extension noted above also explains the tendency of almost all individuals to reject nontrivial symmetric bets. Indeed, the approach presented here can explain most phenomena that Prospect Theory was designed to explain[8] (and a fortiori, more than Friedman-Savage-type theories), but here the explanation is based on reasonable assumptions about the environment rather than on *ad hoc* assumptions regarding an unobservable utility function.

The present approach also generates testable predictions that have no counterpart in received theory. For example, Proposition 3 suggests that agents will not be vulnerable indefinitely to arbitrage ("dutch books"); as

[8] The major exceptions are that Prospect Theory (a) allows a nonlinear weighting function to replace probabilities to account for "certainty effects" and the like, and (b) postulates inconsistencies in a prior "editing" phase to account for some "framing effect" anomalies including "preference reversals" and "aspiration level effects." Possible extensions to cover (a) are noted in fn. 3 above. As for "editing," Leland (private correspondence, 1988) suggests that "framing" anomalies can be explained by extending the present approach to allow the current wealth reference point to be replaced by some other reference point that seems appropriate for a given decision problem. His argument in brief is that the existing value function V will be insensitive in the relevant range for such problems and will be known to lead to poor decisions; but the obvious alternative of "recalibration or reoptimization... is a costly endeavor." Consequently agents will prefer a third alternative: to shift V over to gain maximal sensitivity, which reduces to reference point replacement. Whether or not this idea can be formalized, it is clear that the present approach is no weaker than Prospect Theory in explaining such anomalies.

experience accumulates, the value function's S-shape will tend to relax. Another novel set of implications arise from the dependence of the value function on the distribution F of prospective opportunities. For example, a leftward skew to F will impart a similar skew to V; this suggests that individuals from poorer countries, who are less familiar with large positive values of personal wealth, will tend to appear more risk-averse. In general, the distribution F can differ systematically across populations (or perhaps can be controlled via long training exercises in the laboratory) and so these novel implications seem testable. The evidence cited in Leland (1986) appears to be favorable.

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[Footnotes]

[1] **Rational Choice and the Framing of Decisions**
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