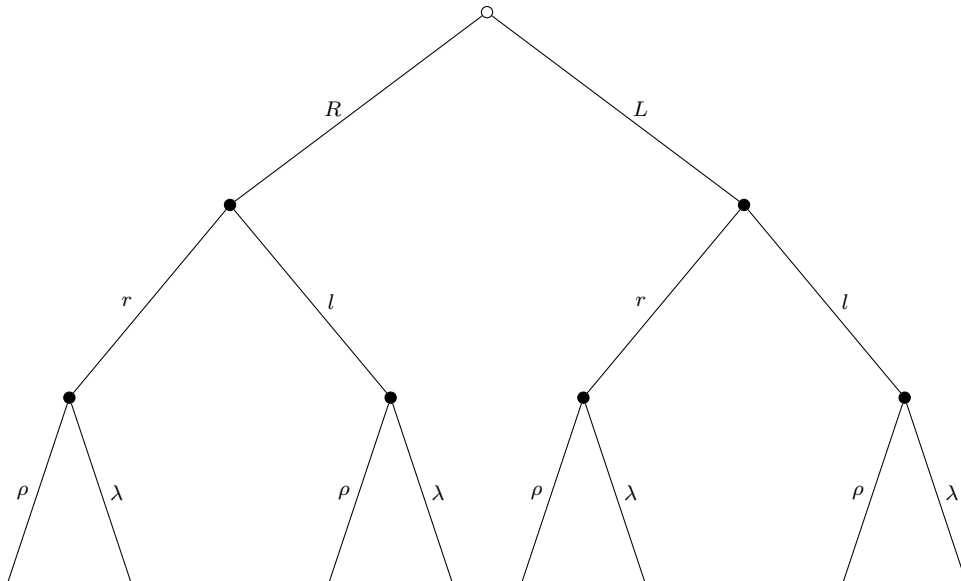


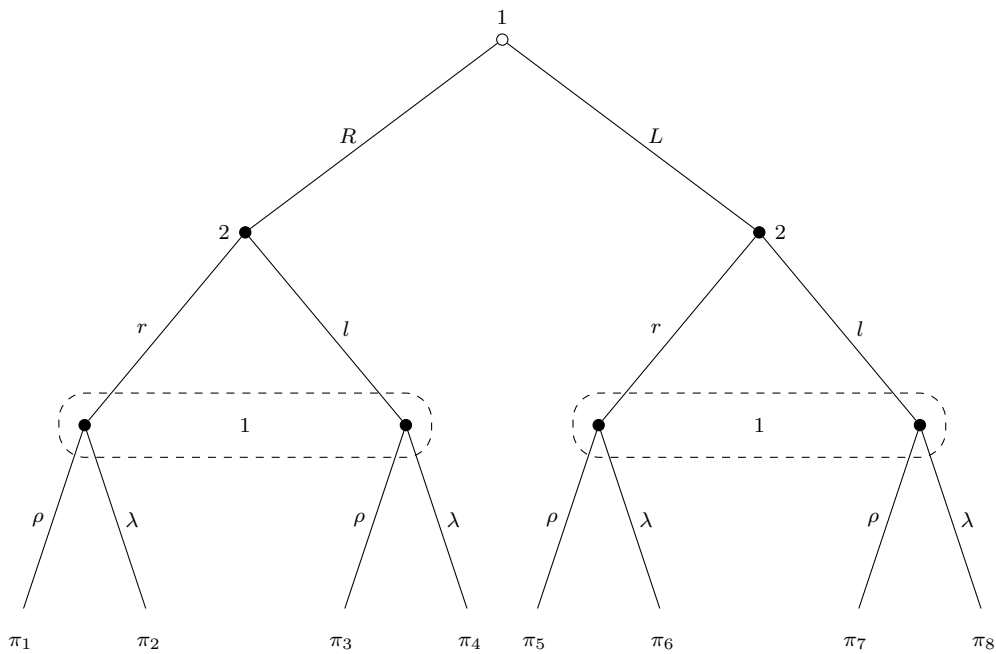
# Part I. Problems

## Problem 1

(a)



(b)



The  $\pi_i$ 's are, of course, payoff vectors in  $R^2$ .

(c)

The complete set of pure strategies for each player is

$$S_1 = \{R\rho\rho, R\rho\lambda, R\lambda\rho, R\lambda\lambda, L\rho\rho, L\rho\lambda, L\lambda\rho, L\lambda\lambda\}$$

$$S_2 = \{rr, rl, lr, ll\}.$$

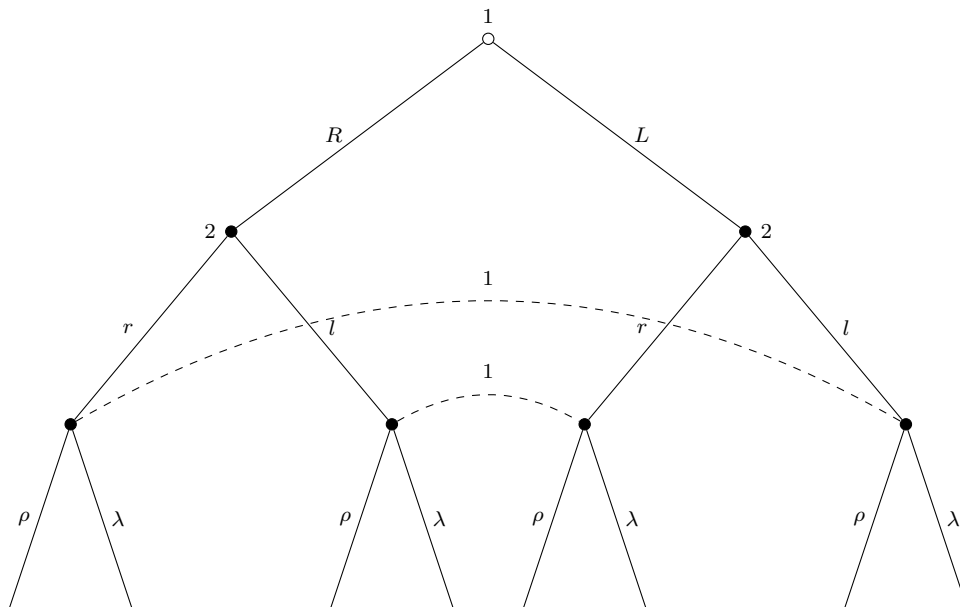
The full normal form is

|          |                        | Player 2  |           |           |           |
|----------|------------------------|-----------|-----------|-----------|-----------|
|          |                        | <i>rr</i> | <i>rl</i> | <i>lr</i> | <i>ll</i> |
| Player 1 | <i>R\rho\rho</i>       | $\pi_1$   | $\pi_1$   | $\pi_3$   | $\pi_3$   |
|          | <i>R\rho\lambda</i>    | $\pi_1$   | $\pi_1$   | $\pi_3$   | $\pi_3$   |
|          | <i>R\lambda\rho</i>    | $\pi_2$   | $\pi_2$   | $\pi_4$   | $\pi_4$   |
|          | <i>R\lambda\lambda</i> | $\pi_2$   | $\pi_2$   | $\pi_4$   | $\pi_4$   |
|          | <i>L\rho\rho</i>       | $\pi_5$   | $\pi_7$   | $\pi_5$   | $\pi_7$   |
|          | <i>L\rho\lambda</i>    | $\pi_6$   | $\pi_8$   | $\pi_6$   | $\pi_8$   |
|          | <i>L\lambda\rho</i>    | $\pi_5$   | $\pi_7$   | $\pi_5$   | $\pi_7$   |
|          | <i>L\lambda\lambda</i> | $\pi_6$   | $\pi_8$   | $\pi_6$   | $\pi_8$   |

The reduced form is

|          |                      | Player 2   |            |
|----------|----------------------|------------|------------|
|          |                      | <i>r</i> · | <i>l</i> · |
| Player 1 | <i>R\rho</i> ·       | $\pi_1$    | $\pi_3$    |
|          | <i>R\lambda</i> ·    | $\pi_2$    | $\pi_4$    |
|          | <i>L</i> · $\rho$    | $\pi_5$    | $\pi_7$    |
|          | <i>L</i> · $\lambda$ | $\pi_6$    | $\pi_8$    |

(d)



The probability of Player 1 choosing  $\rho$  on his final move conditional on being in the “matched” information set is

$$\begin{aligned}
 P(\rho|matched) &= \frac{P(\rho, matched)}{P(matched)} \\
 &= \frac{\sigma_1(R\rho \cdot)\sigma_2(r) + \sigma_1(L \cdot \rho)\sigma_2(l)}{\sigma_1(R\rho \cdot)\sigma_2(r) + \sigma_1(R\lambda \cdot)\sigma_2(r) + \sigma_1(L \cdot \rho)\sigma_2(l) + \sigma_1(L \cdot \rho)\sigma_2(l)},
 \end{aligned}$$

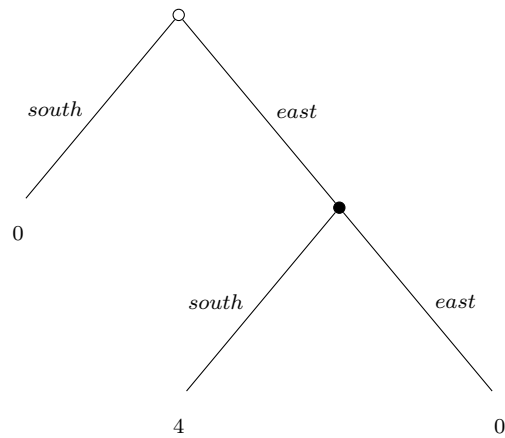
where  $\sigma_i(s)$  denotes the probability that player  $i$  assigns to one of his strategy  $s$ .

(e)

Kuhn’s theorem states that in a game of perfect recall, for any mixed strategy there exists a behavioral strategy with equivalent (expected) payoff. Therefore, player 1 in the game in part (b), where he recalls his previous moves perfectly, has an equivalent behavioral strategy for any mixed strategy. However, this is not necessarily the case in part (d), where player 1 has imperfect recall.

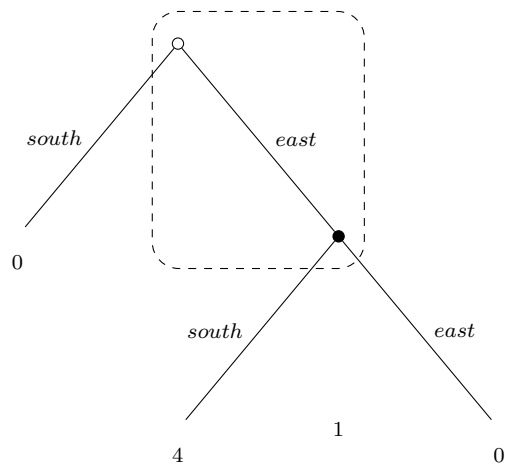
## Problem 2

(a)



The solution is  $(east, south)$ .

(b)



(c)

This is (one-player) game of imperfect recall. As in any game, the player must use the same strategy (either pure or mixed) at each node in the information set. Suppose the player has a mixed strategy and assigns probability  $p$  to turning *south*. Then his expected payoff is

$$\begin{aligned} Eu(p) &= p * 0 + (1 - p) \{ p * 4 + (1 - p) * 1 \} \\ &= 1 + 2p - 3p^2. \end{aligned}$$

The optimal  $p^*$  should satisfy the first order condition

$$2 - 6p^* = 0 \iff p^* = \frac{1}{3}.$$

The corresponding payoff is

$$Eu(p^*) = 1 + \frac{2}{3} - \frac{1}{3} = \frac{4}{3}.$$

**(d)**

The decision rule of turning *south* with a constant probability can be time-inconsistent.  $p^* = 1/3$  is the optimal committed strategy. However, if the player actually uses this strategy, then whenever he comes to a junction the probability that it is the first junction is  $\frac{1}{1+\frac{2}{3}} = 0.6$ . In this case, it would be profitable to deviate to  $\hat{p} = 1$ , since that has expected payoff  $0(.6) + 4(.4) = 1.6 > \frac{4}{3}$ ! Of course, if he actually deviates, then the probability is different... How to resolve this paradox? Philosophers are still arguing over it.

### Problem 3

**(a)**

Suppose player 1 assigns  $p$  to  $H$  and player 2 assigns  $q$  to  $h$ . Then,

$$\begin{aligned} f_1(H, \sigma_2) = f_1(T, \sigma_2) &\iff f_1(H, q * h + (1 - q) * t) = f_1(T, q * h + (1 - q) * t) \\ &\iff q * a + (1 - q) * 0 = q * 0 + (1 - q) * b \\ &\iff q = \frac{b}{a + b} \\ f_2(h, \sigma_1) = f_2(t, \sigma_1) &\iff f_2(h, p * H + (1 - p) * T) = f_2(t, p * H + (1 - p) * T) \\ &\iff p * 0 + (1 - p) * c = p * d + (1 - p) * 0 \\ &\iff p = \frac{c}{c + d}. \end{aligned}$$

Therefore, the unique mixed NE is

$$\left( \frac{c}{c + d} \cdot H \oplus \frac{d}{c + d} \cdot T, \frac{b}{a + b} \cdot h \oplus \frac{a}{a + b} \cdot t \right).$$

**(b)**

$$\begin{array}{ll} \frac{\partial p}{\partial a} = 0 & \frac{\partial q}{\partial a} = \frac{-b}{(a + b)^2} \\ \frac{\partial p}{\partial b} = 0 & \frac{\partial q}{\partial b} = \frac{a}{(a + b)^2} \\ \frac{\partial p}{\partial c} = \frac{d}{(c + d)^2} & \frac{\partial q}{\partial c} = 0 \\ \frac{\partial p}{\partial d} = \frac{-c}{(c + d)^2} & \frac{\partial q}{\partial d} = 0 \end{array}$$

(c)

The intuition of mixed strategy is that a player is indirectly maximizing his payoff by making the other player(s) indifferent among her(their) pure strategies. Therefore, in a neighborhood of the mixed NE, the optimal choice of probabilities  $p$  and  $q$  has no dependence on the players' own payoff structure but on the other player's.

## Problem 4

(a)

We first look for pure NE.  $C$  is dominated by  $L$  and  $R$ , and  $B$  is dominated by  $T$  and  $M$ . Eliminating column  $C$  and row  $B$ , we have

|          |     |                                |                                |
|----------|-----|--------------------------------|--------------------------------|
|          |     | Player 2                       |                                |
|          |     | $L$                            | $R$                            |
| Player 1 | $T$ | $2, 0$                         | $\underline{4}, \underline{2}$ |
|          | $M$ | $\underline{3}, \underline{4}$ | $2, 3$                         |

So the pure NEs are  $\{(M, L), (T, R)\}$ .

Now, we look for a mixed NE. Suppose player 1 assigns  $p$  to  $T$  and player 2 assigns  $q$  to  $L$ . Then,

$$\begin{aligned}
 f_1(T, q * L + (1 - q) * R) = f_1(M, q * L + (1 - q) * R) &\iff q * 2 + (1 - q) * 4 = q * 3 + (1 - q) * 2 \\
 &\iff q = \frac{2}{3} \\
 f_2(L, p * T + (1 - p) * M) = f_2(R, p * T + (1 - p) * M) &\iff p * 0 + (1 - p) * 4 = p * 2 + (1 - p) * 3 \\
 &\iff p = \frac{1}{3}.
 \end{aligned}$$

So the mixed NE is

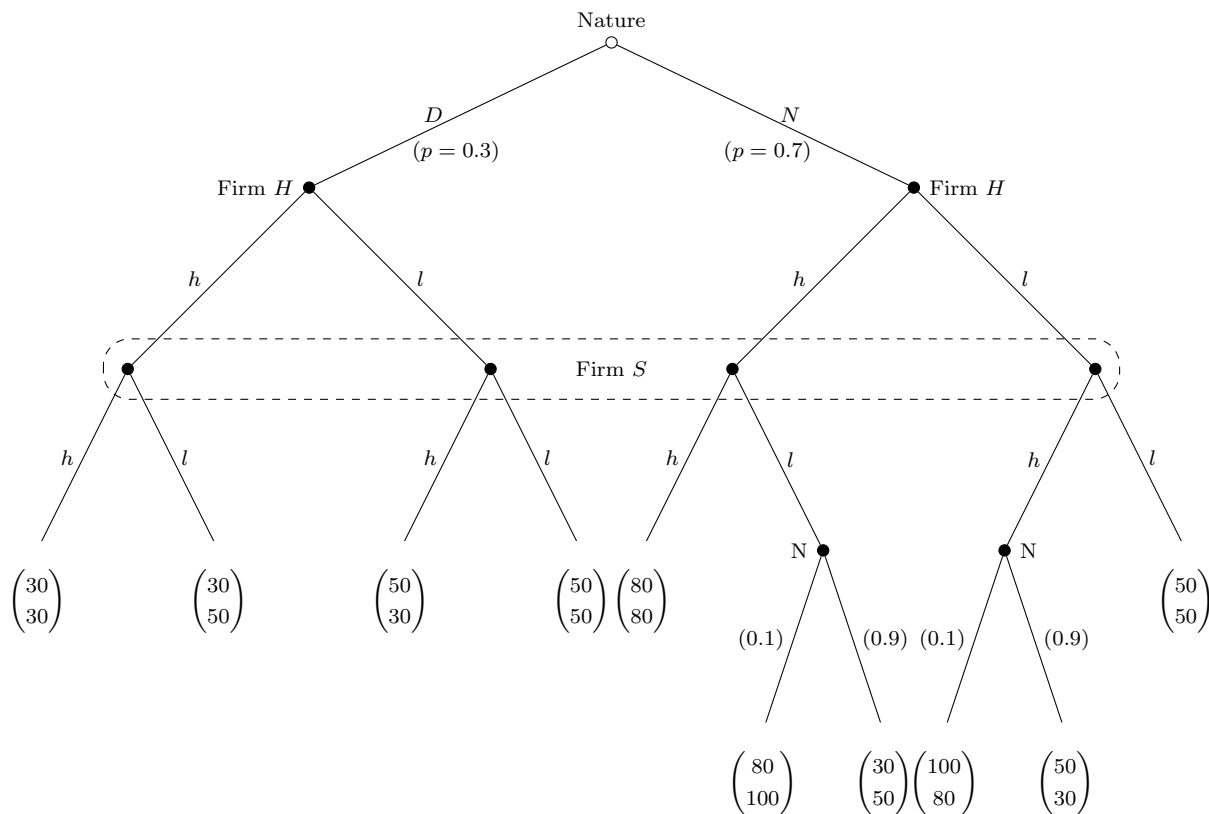
$$\left( \frac{1}{3} \cdot T \oplus \frac{2}{3} \cdot M, \frac{2}{3} \cdot L \oplus \frac{1}{3} \cdot R \right).$$

(b)

Tree omitted to conserve space. Three branches from the initial node (owned by player 1), and each of the resulting three nodes (all owned by player 2) has 3 branches, resulting in 9 terminal nodes, each having the payoff vector from the appropriate cell in the matrix.

### Problem 5

(a)



(b)

The corresponding normal form game is

|        |      | Firm S     |            |
|--------|------|------------|------------|
|        |      | $h$        | $l$        |
| Firm H | $hh$ | 65, 65     | 33.5, 53.5 |
|        | $hl$ | 47.5, 33.5 | 44, 50     |
|        | $lh$ | 71, 65     | 39.5, 53.5 |
|        | $ll$ | 53.5, 33.5 | 50, 50     |

in which, for example, the top-right cell is calculated as

$$.3 * (30, 50) + .7 * \{ .1 * (80, 100) + .9 * (30, 50) \} = (33.5, 53.5).$$

(c)

|        |           |                       |                       |
|--------|-----------|-----------------------|-----------------------|
|        |           | Firm S                |                       |
|        |           | <i>h</i>              | <i>l</i>              |
| Firm H | <i>hh</i> | 65, <u>65</u>         | 33.5, 53.5            |
|        | <i>hl</i> | 47.5, 33.5            | 44, <u>50</u>         |
|        | <i>lh</i> | <u>71</u> , <u>65</u> | 39.5, 53.5            |
|        | <i>ll</i> | 53.5, 33.5            | <u>50</u> , <u>50</u> |

The pure NE are  $\{(lh, h), (ll, l)\}$ .

Note that *hl* and *hh* are dominated by *ll* and *lh*, respectively. Therefore, we have a reduced form:

|        |           |            |            |
|--------|-----------|------------|------------|
|        |           | Firm S     |            |
|        |           | <i>h</i>   | <i>l</i>   |
| Firm H | <i>lh</i> | 71, 65     | 39.5, 53.5 |
|        | <i>ll</i> | 53.5, 33.5 | 50, 50     |

To find the mixed strategy, let us say that firms H and S assign prob.  $p$  and  $q$  to *lh* and *h*, respectively. Then, we have

$$\begin{aligned}
 f_1(lh, q * h + (1 - q) * l) &= f_1(ll, q * h + (1 - q) * l) \\
 \iff q * 71 + (1 - q) * 39.5 &= q * 53.5 + (1 - q) * 50 \\
 \iff q &= \frac{3}{8} \\
 f_2(h, p * lh + (1 - p) * ll) &= f_2(l, p * lh + (1 - p) * ll) \\
 \iff p * 65 + (1 - p) * 33.5 &= p * 53.5 + (1 - p) * 50 \\
 \iff p &= \frac{33}{56}
 \end{aligned}$$

Hence, the mixed NE is  $((\frac{3}{8}, \frac{5}{8}), (\frac{33}{56}, \frac{23}{56}))$ .



## Part II. Textbook problems

### 8.D.9

(a)

Since all the other strategies are too risky and I am fairly risk-averse, I would choose  $M$ .

(b)

|          |     |                                  |                        |                    |                                  |
|----------|-----|----------------------------------|------------------------|--------------------|----------------------------------|
|          |     | Player 2                         |                        |                    |                                  |
|          |     | $LL$                             | $L$                    | $M$                | $R$                              |
| Player 1 | $U$ | $\underline{100}, \underline{2}$ | $-100, 1$              | $0, 0$             | $-100, -100$                     |
|          | $D$ | $-100, -100$                     | $\underline{100}, -49$ | $\underline{1}, 0$ | $\underline{100}, \underline{2}$ |

The pure NE are  $\{(U, LL), (D, R)\}$ .

To find the mixed NE, player 2 has 11 possible subsets with two or more strategy profiles of his strategy set. They are  $\{LL, L, M, R\}, \{LL, L, M\}, \{LL, L, R\}, \{LL, M, R\}, \{L, M, R\}, \{LL, L\}, \{LL, M\}, \{LL, R\}, \{L, M\}, \{L, R\}$ , and  $\{M, R\}$ .

For each possible combination, we set up the system of equations of expected payoffs and check solvability and whether the probabilities are non-negative and sum to 1 for each player. With the help of the software Gambit, we can find that only  $\{LL, L\}$  is the possible part of a mixed NE.

If we suppose player 1 assigns probability  $p$  to  $U$  and player 2 assigns  $q$  to  $LL$ , respectively, then we have

$$\begin{aligned}
 f_1(U, q * LL + (1 - q) * L) &= f_1(D, q * LL + (1 - q) * L) \\
 \iff q * 100 + (1 - q) * (-100) &= q * (-100) + (1 - q) * 100 \\
 \iff q &= \frac{1}{2} \\
 f_2(LL, p * U + (1 - p) * D) &= f_2(L, p * U + (1 - p) * D) \\
 \iff p * 2 + (1 - p) * (-100) &= p * 1 + (1 - p) * (-49) \\
 \iff p &= \frac{51}{52}
 \end{aligned}$$

Therefore, the mixed NE is  $\left(\frac{51}{52}U \oplus \frac{1}{52}D, \frac{1}{2}LL \oplus \frac{1}{2}L\right)$ .

(c)

$M$  is a rationalizable strategy. For example, if I conjecture that player 1 plays a mixed strategy of  $p = 1/2$ , then I can justify  $M$  as the unique best response to it.

(d)

If preplay communication were possible, I would try to make an agreement to play one of the Pareto dominant solutions, which are  $\{(U, LL), (D, R)\}$ . Therefore, I would play either  $LL$  or  $R$ , depending

on the agreement.

### 8.D.5

(a)

Denote  $0 \leq s_1, s_2 \leq 1$  as the locations that vendors 1 and 2 choose. Given the information in the problem, the payoff functions are

$$f_1(s_1, s_2) = \begin{cases} \frac{s_1+s_2}{2} & \text{if } s_1 < s_2 \\ \frac{1}{2} & \text{if } s_1 = s_2 \\ 1 - \frac{s_1+s_2}{2} & \text{if } s_1 > s_2 \end{cases}$$
$$f_2(s_1, s_2) = \begin{cases} 1 - \frac{s_1+s_2}{2} & \text{if } s_1 < s_2 \\ \frac{1}{2} & \text{if } s_1 = s_2 \\ \frac{s_1+s_2}{2} & \text{if } s_1 > s_2 \end{cases}$$

If  $s_1 < s_2$ , both vendors have incentive to move closer to the other vendor to steal more customers. The case of  $s_1 > s_2$  allows the same incentives for deviation. Therefore,  $s_1 = s_2$  is a necessary condition for a Nash equilibrium.

If  $s_1 = s_2 < 1/2$  or  $s_1 = s_2 > 1/2$ , both players have incentive to move closer to the midpoint, since a slight deviation will allow for more customers. Therefore,  $s_1 = s_2 = 1/2$  is the unique pure Nash equilibrium.

(b)

With three players, there are three cases:

- (i) If  $s_i = s_j = s_k$ , each vendor has incentive to move right or left.
- (ii) If  $s_i = s_j \neq s_k$ , vendor  $k$  has incentive to move closer to  $i$  and  $j$ .
- (iii) If  $s_i \neq s_j \neq s_k$ , both vendors not in the middle have incentives to move farther from the middle.

Hence, there exists no pure Nash equilibrium.