4. Preferences and Demand

See Varian Chapters 7-9

I. Preference Relations and Utility Functions

A. Our main goal in this section is to model how people choose among consumption opportunities.

B. Each opportunity is called a **bundle**.

C. A bundle is represented as a vector $\mathbf{x} \in \mathbf{R}^n_+$ with dimension n equal to the number of goods. On Amazon, $n > 100,000$.

D. But the main ideas can be conveyed with $n = 2$, where $\mathbf{x} = (x_1, x_2)$.

- $\mathbf{x} = (7, 4)$ is the bundle consisting of 7 units of good 1 and 4 units of good 2.
- E.g., $x_1 = \#$ of cowboy wraps at lunch truck, and $x_2 = \#$ sodas.
- Or, $x_1 =$ number of useful ideas in the lecture, and $x_2 =$ style points.

Ex: Bundles in a 2-dimensional consumption space $\mathbf{R}^2_+$.

E. Preferences Over Bundles

1. We assume that people can compare alternative bundles, and know which they like better.
2. A **preference** is a relation between between pairs of bundles.
3. Taking a consumer's response to any two bundles $\mathbf{x}$ and $\mathbf{y}$, we write
   - $\mathbf{x} \sim \mathbf{y}$ ("indifference") if she is just as happy with $\mathbf{x}$ as she is with $\mathbf{y}$.

     OR

- $\mathbf{x} \succ \mathbf{y}$ or $\mathbf{x} \prec \mathbf{y}$ ("strict preference") if she either happier with $\mathbf{x}$ than $\mathbf{y}$ or is happier with $\mathbf{y}$ than $\mathbf{x}$.
- $\mathbf{x} \succeq \mathbf{y}$ (or $\mathbf{x} \preceq \mathbf{y}$) means that this consumer is either indifferent between $\mathbf{x}$ and $\mathbf{y}$ or strictly prefers $\mathbf{x}$ to $\mathbf{y}$ (or she is either indifferent between y and $\mathbf{x}$ or strictly prefers $\mathbf{y}$ to $\mathbf{x}$). This is called **weak preference** or just plain **preference**.

Ex: Indifference curves (IC's) and better sets in a 2-dimensional consumption space.

4. Preferences are defined by three properties:

   a. Complete: Either $\mathbf{x} \succeq \mathbf{y}$, or $\mathbf{y} \succeq \mathbf{x}$, or both (in which case $\mathbf{x} \sim \mathbf{y}$).

   b. Reflexive: $\mathbf{x} \succeq \mathbf{x}$.

   c. Transitive: If $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{z}$ then $\mathbf{x} \succeq \mathbf{z}$.

   Ex: Preference no-no: Indifference curves that cross.

5. Preferences are often assumed to have additional properties. Three of the most important are:

   a. Monotone (increasing): More is better: $x_i \geq y_i \;\; \forall i \implies \mathbf{x} \succeq \mathbf{y}$.

   b. Convex: If $\mathbf{x} \sim \mathbf{y}$ then for any $0 \leq \alpha \leq 1$,

   $(\alpha x_1 + (1 - \alpha y_1), \alpha x_2 + (1 - \alpha y_2)) \succeq (x_1, x_2)$ (Mixtures are better)

   c. Inada: $[x_i > 0 \;\; \forall i]$ & $[y_i = 0$ for some $i] \implies \mathbf{x} \succ \mathbf{y}$.

   Ex: Non-monotone prefs. Non-convex prefs. Non-Inanda prefs.

F. Utility Functions.

The preference theory we just talked about seems cumbersome — you wouldn't want to list all possible pairs, their relation, and check whether the properties hold ! So it is very nice that we have utility functions to work with.

- If preferences (which are automatically complete, reflexive and transitive) are also continuous (a property awkward to define formally, but intuitively clear) and monotone, then it is known that we can represent those preferences by a continuous utility function. [Varian, p.97 sketches a proof.]
- A **utility function** u assigns a real number $u(\mathbf{x})$ to each bundle $\mathbf{x}$.
- The utility function u **represents** preferences $\succeq$ if
  1. $\mathbf{x} \succ \mathbf{y}$ if and only if $u(\mathbf{x}) > u(\mathbf{y})$, and
  2. $\mathbf{x} \sim \mathbf{y}$ if and only if $u(\mathbf{x}) = u(\mathbf{y})$.
- So a utility function is just like a production function whose inputs are bundles and whose output is the degree of satisfaction. From a mathematical perspective, the only difference is that:
  1. the *same* preferences can be represented by several *different* utility functions. More specifically,
  2. Any monotone increasing transformation v of a utility function u represents the same preferences as u. As long as $v(\mathbf{x}) = h(u(\mathbf{x}))$ for some strictly increasing function h, then you can use either u or v to represent those preferences, whichever is more convenient.
  3. Here's why. In switching from u to v, you don't change the shape of any IC or their ordering. You only change the labels on the IC's from $u =$ const to $v =$ some other constant. (Don't do this with production functions !!)
  4. Can generally choose smooth(continuously differentiable)utility functions.

- The partial derivative of a utility function is called its **marginal utility**: $mu_i \equiv \frac{\partial u}{\partial x_i}$. It depends on the choice of a utility function, but the next construct is the same for any utility function representing a particular set of preferences.

G. The **marginal rate of substitution** between two goods i and j is

$$MRS_{ij} \equiv -\frac{dx_j}{dx_i}|_{u=const} = \frac{mu_j}{mu_i} \equiv \frac{\partial u}{\partial x_j} / \frac{\partial u}{\partial x_i}. \tag{1}$$

1. MRS_{ij} is the consumer's trade-off rate between those two goods. It is how many (micro)units of i that the consumer requires to just compensate for losing one (micro)unit of j.
2. MRS_{12} is just the slope of the indifference curve in (x_1, x_2) space.
3. In higher dimensions, there is an indifference hypersurface, and there MRS_{ij} is the slope of that surface in the $j - i$ plane.
4. MRS_{ij} is invariant to the choice of utility function to represent given preferences! (If you like, you can check this by taking the ratio of partial derivatives of $v(\mathbf{x}) = h(u(\mathbf{x}))$ and noting that the h' in the numerator cancels the h' in the denominator.)
5. The Inada property mentioned earlier implies that $MRS_{ij}(\mathbf{x}) \to \infty$ as $x_i \to 0$, and $MRS_{ij}(\mathbf{x}) \to 0$ as $x_j \to 0$ The idea is that IC's don't intersect axes when all goods are essential in the sense that marginal utility of each good gets arbitrarily large when the person has almost none of it.

Ex: Perfect substitutes: $u(x_1, x_2) = x_1 + cx_2$. Then $MRS_{12}(x_1, x_2) = c > 0$; the tradeoff rate is constant.

Ex: Cobb-Douglas utility: $u(x_1, x_2) = \ln x_1 + c \ln x_2$. Then $MRS_{12}(x_1, x_2) = \frac{cx_1}{x_2} > 0$. Convex, Inada. What can we say about $v(x_1, x_2) = \exp(u(x_1, x_2))$?

Ex: CES utility: $u(x_1, x_2) = \frac{1}{\rho} \ln(x_1^\rho + cx_2^\rho)$, where $\rho \in (-\infty, 1]$. Can show that this nests the previous two cases, which correspond respectively to $\rho = 1, 0$. Generalizes directly to more than 2 goods. Very useful in applied work.

Ex: Quasilinear utility: $U(x_0, x_1) = x_0 + g(x_1)$. Think of good 0 as money, or purchasing power. $MRS_{ij}(x_0, x_1) = g'(x_1)$. Very very useful in applied work.

II. The Direct Consumer Problem and (Marshallian) Demand

Now we can model choice and see where demand functions come from. Let me follow textbooks for the next hour. Later I may show you a streamlined approach (with roots in Marshall) that I developed with Jozsef Sakovics.

Suppose that the consumer is constrained only by available income and by the prices of the goods.

A. Let m be the available money to spend, while p_1 the price of good 1, p_2 the price of good 2 etc.

B. Then the **budget constraint** is

$$m \geq \mathbf{p} \cdot \mathbf{x} = p_1 x_1 + p_2 x_2 + ..., \tag{2}$$

(i.e. you can't spend more than you have.)

C. If she has strictly monotone preferences, a consumer will spend all of her money — putting money in a savings account could count as one of the x_i's, and m as such brings no utility. So we can safely assume that equation (1) holds as an equality:

$$m = \mathbf{p} \cdot \mathbf{x} = p_1 x_1 + p_2 x_2 + ... \tag{3}$$

Ex: Budget constraints and budget sets.

D. Then we have the following constrained optimization problem.

$$\max u(x_1, x_2, ...) \quad \text{s.t. } m = p_1 x_1 + p_2 x_2 + ... \tag{4}$$

We form the Lagrangian

$$\mathcal{L} = u(x_1, x_2, ...) + \lambda(m - p_1 x_1 + p_2 x_2 + ...) \tag{5}$$

Differentiating, we get first order conditions:

$$\begin{aligned} 0 &= \frac{\partial u}{\partial x_1} - \lambda p_1 \implies mu_1 = \lambda p_1 \\ 0 &= \frac{\partial u}{\partial x_2} - \lambda p_2 \implies mu_2 = \lambda p_2 \\ 0 &= ... \\ 0 &= \frac{\partial \mathcal{L}}{\partial \lambda} \implies m = p_1 x_1 + p_2 x_2 + ... \end{aligned} \tag{6}$$

E. Simultaneously solving these FOCs gives us optimal consumption decisions, at least if standard assumptions hold. (Namely, strong monotonicity, smooth indifference curves, Inada and convexity ensure that (6) has a unique solution that solves the consumer choice problem (4). Even without Inada and convexity one can often get an interior solution to (4) where (6) holds.).

F. Without actually solving the FOCs (6), dividing condition i by condition j gives us a familiar expression:

$$p_i/p_j = \frac{\partial u}{\partial x_i} / \frac{\partial u}{\partial x_j} = MRS_{ji}. \tag{7}$$

G. That is, the budget line and indifference curve have the same slope at the optimal bundle and, given the last condition in (6), they are actually tangent at that point.

Ex: An example with Cobb-Douglas preferences.

Ex: Direct consumer problem with indifference curves.

Ex: Corner solutions. Non-convex prefs. non-monotone prefs.

H. Solving problem (4), sometimes called the *direct consumer problem*, gives us the quantity x_i demanded for each good i as a function of prices and income. This is the individual's **Marshallian demand** $x_i^*(\mathbf{p}, m)$.

III. Comparative statics of Marshallian demand.

If we hold the price of other goods and income constant in $x_i^*(p_1, p_2, ..., m)$, we get (for that individual consumer) the old demand curve we studied in the first week $x_i(p_i)$. It shifts with changes in other prices and changes in income.

Ex: Marshallian demand from Cobb-Douglas preferences. Obtain

$$\ln x_i = A + e_{ii} \ln p_i + e_{ij} \ln p_j + \eta_i \ln m \tag{8}$$

with $e_{ii} = -1, e_{ij} = 0, \eta_i = 1$, and $A = \ln$ expenditure share.

A. Income sensitivity: $\frac{\partial x_i^*}{\partial m}$

We just saw that m affects the quantity demanded. What happens when income (actually, expenditure) changes? Of course, it depends on the structure of preferences – we'll go through a few of the possible cases, drawing the Income Expansion Paths (IEPs).

- **Normal goods** are goods that consumers buy more of as income increases: $\frac{\partial x_i^*}{\partial m} > 0$.
- **Inferior goods** are goods that consumers buy less of as income increases: $\frac{\partial x_i^*}{\partial m} < 0$.
- With **quasilinear** preferences and sufficient income, consumers don't change the amount they buy as income increases: $\frac{\partial x_i^*}{\partial m} = 0$.

Ex: To explain that last point, suppose that $u(x_0, x_1) = x_0 + g(x_1)$. Check that all extra money gets spent on good 0 (i.e., kept as cash) and none on good 1, once enough x_1 has been purchased so that $g'(x_1) \leq p_1$.

B. If preferences are **homothetic**, then IEPs are straight lines, so the proportion of total consumption represented by each good remains the same regardless of income.

1. If m doubles, then consumption of each good doubles.
2. If you multiply m by t, then consumption of each good gets multiplied by t
3. unit income elasticity: $\frac{\partial \ln x_i^*}{\partial \ln m} = 1$.

Ex: Cobb-Douglas preferences are homothetic.

C. Price Changes: $\frac{\partial x_i^*}{\partial p_i}$.

Marshallian demand is more complicated than you might think when it comes to the basic question of own-price effects. We'll go through the messy (but standard) approach now, and later will see how other versions of demand make things more intuitive.

Two different effects are at work (sometimes in opposite directions):

- **Substitution effect**: A change in the price ratio affects consumers' trade-off rates among goods.
- **Income effect**: The consumer's spending power (her real income) changes because goods overall are cheaper (if the price went down) or more expensive (if the price went up).

1. Income Effect. As we just saw, the income effects of price changes can be either positive or negative depending on whether the price increase or decreases and whether the good is normal or inferior.
2. Substitution Effect. You can isolate the substitution effect by imagining what a consumer would choose if the price change happened and then we either gave the consumer some money (if the price increased) or took away some money (if the price decreased) to just compensate for the change in purchasing power.
3. There are several slightly different methods to separate the substitution effect from the income effect; we'll just do one, called the **Hicks decomposition**.
4. What if we add or subtract just enough income so that the consumer's optimal consumption after the price change, though different, is just as pleasing to him as before?
5. The change in consumption due to the change in price *after* so compensating her income is a measure of the substitution effect.

   Ex: Graphical example: roll and shift.

6. Hicksian Demand

   - There is an alternative version of the demand function called **Hicksian demand** and denoted $h_i(\mathbf{p}, u)$, that consists only of this compensated demand.
   - It is defined as the quantity of the good a consumer demands when choosing the cheapest (expenditure minimizing) bundle she could buy while maintaining a given level u of utility.
   - Hicksian demand charts only changes in quantity demanded that are due to substitution effects.
   - Hicksian demand is always inversely related to price, i.e., satisfies $\frac{\partial h_i(\mathbf{p},u)}{\partial p_i} < 0$.

7. **The Slutsky Equation** formally shows how to decompose $(\frac{\partial x_i^*}{\partial p_i})$ into an income and substitution effect. Below $e(\mathbf{p}, u)$ is the minimum expenditure m necessary to acquire a bundle that brings utility level u given price vector $\mathbf{p}$. It is an easy consequence of the envelope theorem that $\frac{\partial e}{\partial p_i} = x_i^*$.

   - Start with the identity $h_i(\mathbf{p}, u) = x_i^*(\mathbf{p}, e(\mathbf{p}, u))$.

     Differentiate both sides with respect to p_i and rearrange terms to get the Slutsky equation

$$\frac{\partial x_i^*(\mathbf{p}, m)}{\partial p_i} = \frac{\partial h_i(\mathbf{p}, u)}{\partial p_i} - \frac{\partial x_i^*}{\partial m} x_i^*(\mathbf{p}, m). \tag{9}$$

   - There are two terms on the right hand side (RHS) of the Slutsky equation:

     a. The first term is the substitution effect. It is the derivative of Hicksian demand evaluated at $u =$ the maximum level of utility achievable given the prices and income.

b. The second term $-\frac{\partial x_i}{\partial m} x_i(\mathbf{p}, m)$ is the income effect. It turns out to be the income sensitivity found in the previous section weighted by the quantity currently demanded.

8. **The "Law of Demand"** states that $\frac{\partial x_i^*(\mathbf{p},m)}{\partial p_i} < 0$, i.e., that demand curves slope downwards. It is true for all normal goods. Why?

   - The substitution effect is always negative as we have just seen, so the first term on the RHS of (9) is negative.
   - If the good is normal, then the second term on the RHS of (9) is also negative, so the entire RHS is negative, as claimed.

9. Giffen Goods

   a. Usually substitution effects are greater than income effects and so, even if goods aren't normal, demand still slopes downward.

   b. However, it is theoretically possible for a good to be so inferior (over some range of $(\mathbf{p}, m)$) that the income effect overcomes the substitution effect and demand slopes upward.

   c. These theoretical constructs are called **Giffen goods**.

   Ex: Graphical example of a Giffen good.

IV. Duality and other tools.

Pursuing the analogy to the supply side, we can construct functions dual to demand and utility. It is conceptually helpful, though perhaps not quite as important in empirical work.

A. **Indirect utility** is $v(\mathbf{p}, m) = [\max u(\mathbf{x}) \text{ s.t. } \mathbf{p} \cdot \mathbf{x} \leq m] = u(\mathbf{x}^*(\mathbf{p}, m))$.

B. Dual problem to utility max is expenditure min:

$$\min_{\mathbf{x}} \mathbf{p} \cdot \mathbf{x} \text{ s.t. } u(\mathbf{x}) \geq u_o. \tag{10}$$

C. Solution to (10) is called expenditure function, denoted $e(\mathbf{p}, u_o)$, and argmax's are called Hicksian demands, denoted $h_i(\mathbf{p}, u_o)$, used earlier.

Ex: indirect utility, expenditure and Hicksian demand from Cobb-Douglas preferences.

D. see Varian for a bunch of general properties and identities, some of which can help in applied work.

E.g., $h_i(\mathbf{p}, u_o) = x_i^*(\mathbf{p}, e(\mathbf{p}, u_o))$, interpreted as compensated demand, as we will see shortly.

E. **Elasticity form of the Slutsky equation**: Multiply both sides of equation (9) by $\frac{p_i}{x_i}$, also multiply the last term by $\frac{m}{m}$, and simplify, using the usual expressions for elasticities. Writing $s_i = \frac{p_i x_i}{m}$ as the expenditure share of good i we get

$$\epsilon_i = \epsilon_i^h - s_i \epsilon_m. \tag{11}$$

where ϵ_i is the usual own-price elasticity, ϵ_i^h is the Hicksian (or income-compensated or pure substitution) elasticity, and ϵ_m is income elasticity.

This equation is useful because you may have estimates of some of these elasticities and want to know others, which you can get from equation (11).

F. **Roy's Identity** shows how you can recover the ordinary (Marshallian) demand function from the indirect utility function $v(\mathbf{p}, m)$. It says

$$x_i^*(\mathbf{p}, m) = \frac{-\frac{\partial v}{\partial p_i}}{\frac{\partial v}{\partial m}} \tag{12}$$

See Varian p. 106-8 for three (!) proofs and some general remarks.

G. **Shepherd's Lemma** tells us how to recover the Hicksian (income compensated) demand functions from the expenditure function. (Recall an analogous expression by the same name on the supply side.) The demand side version is

$$h_i^*(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_i}. \tag{13}$$

H. **Elasticities identity.** There is a useful formula that connects the values of the various demand elasticities. It is based on a formula discovered by the mathematician Leonhard Euler (1707-1783). The formula applies to homogeneous functions, i.e., functions that for all $\mathbf{y}$ and all $a > 0$ satisfy the identity

$$f(a\mathbf{y}) = a^k f(\mathbf{y}) \tag{14}$$

for some nonnegative integer k. For example, Cobb-Douglas functions are homogeneous of degree k = sum of exponents. Euler showed that any such function can be written

$$y_1 \frac{\partial f}{\partial y_1} + ... + y_n \frac{\partial f}{\partial y_n} = kf(y_1, ..., y_n). \tag{15}$$

It is easy to verify (15) if you wish, simply by totally differentiating both sides of equation (14) with respect to a and evaluating at $a = 1$.

Note that Marshallian demand functions $x_i^*(p_1, ..., p_n, m)$ are homogeneous of degree 0: doubling all prices and income, for example, will have no effect on the optimal quantities of goods purchased, as you can see by inspecting the budget constraint. Applying (15) with $k = 0$ to $x_i^*(p_1, ..., p_n, m)$, we get

$$p_1 \frac{\partial x_i^*}{\partial p_1} + ... + p_n \frac{\partial x_i^*}{\partial p_n} + m \frac{\partial x_i^*}{\partial m} = 0. \tag{16}$$

Denote price elasticities by $e_{ij} = \frac{\partial x_i^*}{\partial p_j} \frac{p_j}{q_i}$ — if $i = j$ this is called the own-price, otherwise the cross-price elasticity of demand — and income elasticity by $\eta_i =$

$\frac{\partial x_i^*}{\partial m}\frac{m}{q_i}$. Now divide all terms in (16) by the quantity demanded $q_i = x_i^*$ to obtain the desired identity

$$e_{i1} + ... + e_{in} + \eta_i = 0, \quad i = 1, ..., n. \tag{17}$$

That is, for any good i, the sum of its income elasticity and own price and all cross price elasticities are zero! You can impose this constraint directly on estimated log-linear demand functions to obtain more accurate results. Another way to think of it is that own price elasticity is interesting to the extent that it differs from -1.0, cross price from from 0 and income elasticity from 1.0.

I. Moneysworth demand. See "Tractable consumer choice," by D. Friedman, Daniel and J. Skovics, *Theory and Decision* 79:2 pp.333–358 (2015).

- Recall that (aside from corner solutions) Marshallian demand is defined by the FOCs in (6). All except the last say that the marginal utility for each good is equal to λ times the price of that good. The last equation is the budget constraint.
- The equations for Hicksian demand are exactly the same except that the budget constraint is replaced by a utility constraint, $u(x_1, x_2, ...) = u_o$.
- Moneysworth demand also uses the same equations except that the budget constraint is replaced by an equation that says λ is the marginal value of keeping some cash for later purchases.
- The article above argues that this version of demand is simpler than either the Marshallian or Hicks versions. It gets rid of income effects, ties nicely to sequential decisions (we will study those later in the course), can handle indivisible goods (where you can't continuously adjust the amount) and liquidity

constraints.

- The article also argues that moneysworth demand is more intuitive and realistic. People usually compare the satisfaction they get per unit of goods purchased to the value they expect to get for their money elsewhere. This can lead to behavior like money illusion and house-money effects that is realistic but inconsistent with Marshallian demand theory.
- As a practical matter, when you estimate an empirical demand function, the moneysworth perspective might help you find better proxies for the income variable.

J. Corners and notches. These notes emphasized interior solutions to the standard (or dual) consumer choice problem. Of course, a choke price p_i^c represents a point above which $\mathbf{x}^*$ is at the $x_i = 0$ corner (or face).

Sometimes additional constraints on consumer choice can be analyzed using only slight modifications. For example, the opportunity set is a natural modification of the budget set when there is a ceiling or floor on the amount purchased, or a subsidy or tax or matching grant.

Ex: Opportunity set with rationing (ceiling).