9. Intertemporal Equilibrium Theory

Not covered in Varian, although Chs 17, 19 and 20 have tangentially related material.

Good references include:

Price Theory and Applications, 6th Ed, Ch 14, by J. Hirshleifer

Financial Theory and Corporate Policy, 3rd Ed, Ch 1-2, by T. Copeland.

I. Overview

A. These notes cover the **general equilibrium** theory of production and exchange **across time**,

1. originally developed by Irving Fisher (top American economist) 100 years ago.
2. Sorts out many of the fundamentals of finance – investment, savings, real vs nominal interest rates, productivity, thrift, etc. – in a simple abstract setting.
3. (The same abstract GE setting can be re-interpreted to explain the basics of international trade!)

B. Finance is that branch of economics that deals with risky trades over time. As such, it has three pillars:

1. Choice and market trades over time – our focus in these notes.
2. Static choice and markets for risky assets – the focus of most MBA-oriented finance texts.
3. Markets for information, typically asymmetric – the focus of ongoing academic research.

C. Deep understanding of the fundamentals is essential when conditions change and established rules-of-thumb become questionable.

D. Most MBA-oriented textbooks skip the first pillar because it is hard for non-economists to understand, and they focus instead on the second pillar.

II. Begin with a two-date barter model, later extend to many dates and monetary exchange.

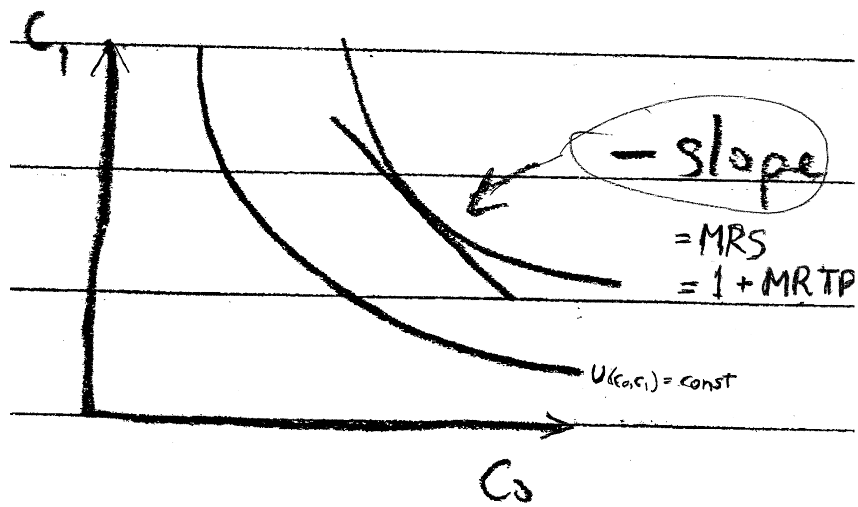


Figure 1:

A. There are two dates: $t = 0$ ("now") and $t = 1$ ("later").

B. There is only one good, c ("corn" or "consumption baskets").

C. There are N agents with given preferences and production opportunities, all with access to a financial market. We now specify these elements.

III. **Preferences** of agent i (index suppressed) are represented by a utility function $U(c_0, c_1)$.

A. Assume classical properties – smooth, strictly monotone, concave, Inada.

B. Inada implies that some consumption at each date is essential, and monotone implies that more consumption is always better.

$$-\text{slope of IC} = \frac{\partial_0 U}{\partial_1 U} = MRS_{01} = 1 + MRTP. \tag{1}$$

C. Most of equation (1) is familiar from an earlier unit. I'm using the shorthand notation $\partial_t U = \frac{\partial U}{\partial c_t}$ for the marginal utilities from current and future consumption, $t = 0, 1$. See Figure 1.

D. The last expression in (1) defines MRTP, the marginal rate of time preference. Equivalently, $\text{MRTP} = \frac{\partial_0 U}{\partial_1 U} - 1$.

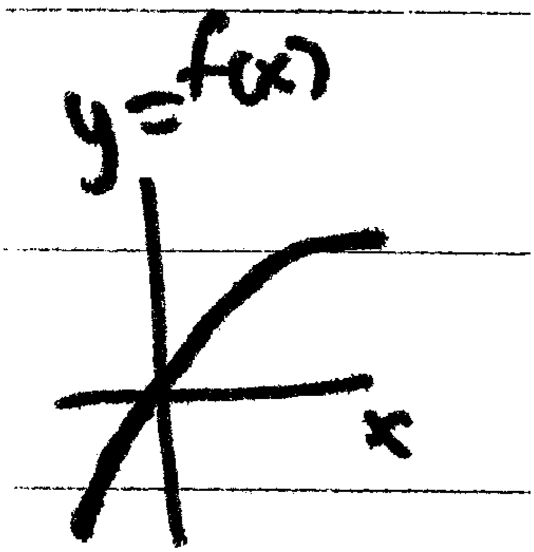


Figure 2: The production function

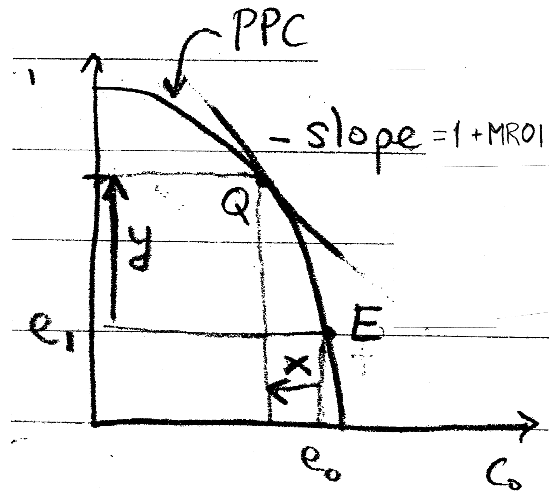


Figure 3: The PPF obtained from that production function by mapping the Origin to E and reversing the x-axis.

E. That is, on the margin, agent i needs MRTP *extra* units of future consumption per unit of foregone current consumption. This turns out to be a more convenient way to express the tradeoff than MRS, which is the *total* number of units of future consumption (the return of the original unit plus the MRTP) required per unit of foregone current consumption.

IV. **Productive opportunities** of agent i (index still suppressed) are represented by a

A. production function $y = f(x)$ as in Figure 2 that relates

B. increments of future consumption $y = \Delta c_1$ to

C. the amount of foregone current consumption $x = -\Delta c_0$.

D. Think of production as planting seed corn, or more generally think about allocating overall resources between immediate consumption and building for the future.

E. As shown in Figure 3, the increments are taken relative to a given endowment point $E = (e_0, e_1)$.

F. We make the standard assumptions that $f(0) = 0, f' > 0, f'' < 0$.

G. The figure shows the corresponding production possibility frontier, $\text{PPF} = \{(q_0, q_1) : \; q_0 = e_0 - x \geq 0, \; q_1 = e_1 + f(x) \geq 0\}$.

Handy summary descriptions are average and marginal return on investment:

$$ROI = f(x) - x, \; AROI = \frac{f(x)}{x} - 1, \; MROI = f'(x) - 1. \tag{2}$$

Or, in more geometrical terms,

$$-\text{slope of PPF} = MRT = 1 + MROI = f'(x). \tag{3}$$

V. A **financial market** is available to all agents at date $t = 0$, in which

A. units of current consumption c_0 can be traded for

B. claims (i.e., **promises**) to units of future consumption c_1.

C. The -slope of the budget line is the price ratio, which we write out as

$$-slope = \frac{p_0}{p_1} = 1 + r. \tag{4}$$

D. Presumably the price p_1 of a promise to 1 unit of c next period is less than the price p_0 of 1 unit of c held now.

E. Thus the price ratio exceeds 1.0. The degree of excess, $r \geq 0$, is called the **real interest rate**.

F. As shown in Figure 4, each unit of c_0 exchanges for $1+r$ units of c_1 in the financial market. To summarize in a diagram,

$$c_0 \xrightarrow{1+r} c_1$$

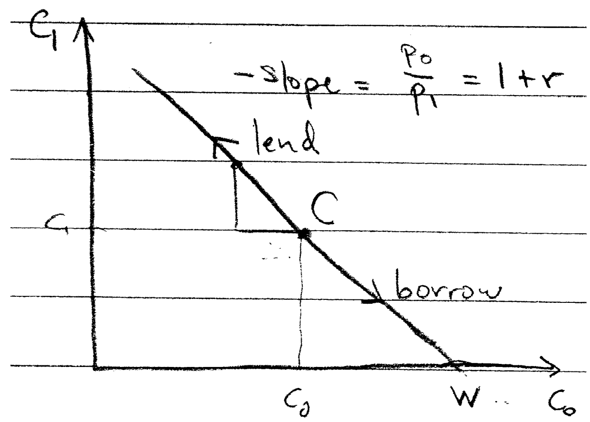


Figure 4: A trader who brings bundle $C = (c_0, c_1)$ to the financial market can trade to obtain any bundle along the budget line. The trader is borrowing [lending] if the post-trade holding of current consumption [future consumption] exceeds the pretrade value.

VI. Results: Wealth and Present Value

A. Given a real interest rate $r > 0$ and consumption stream $C = (c_0, c_1)$, define the **present value** of C as the horizontal axis intercept w of the budget line thru C.

B. High school geometry enables us to compute $w = PV_r(C)$.

- Consider the right triangle with base $[c_0, w]$ on horizontal axis and vertex C.
- We need to compute the length z of the triangle's base in terms of the height c_1 and rise / run = -slope = $1 + r$.
- Thus $1 + r = \frac{c_1}{z}$, so $z = \frac{c_1}{1+r}$.
- We conclude that $w = PV_r(C) = c_0 + z = c_0 + \frac{c_1}{1+r}$.

C. The intuition is that the present value is the amount of maximum amount of present consumption we can get in the financial market. It simply adds the amount c_0 of current consumption we already have to the amount $z = \frac{c_1}{1+r}$ we can get from the future part of our consumption stream.

D. In the static context, the horizontal intercept of the budget line (normalizing that price to 1) is called income. In our intertemporal context, that intercept is instead called **wealth**.

E. Hence we write $w = PV_r(C) = c_0 + \frac{c_1}{1+r}$.

VII. The next result characterizes the agent's optimum investment, given her production function f, her endowment $E = (e_0, e_1)$ and the real interest rate r.

- For reasons elaborated below, an individual agent seeks to maximize wealth by choosing the amount x of first period consumption to invest.
- Formally, her choice problem is

$$\max_x w = PV_r(Q) = q_0 + \frac{q_1}{1+r} = e_0 - x + \frac{e_1 + f(x)}{1+r}. \tag{5}$$

- The FOC is

$$\begin{aligned} 0 = \frac{dw}{dx} &= -1 + \frac{f'(x)}{1+r}. && (6) \\ \Longrightarrow 1 + r &= f'(x) = 1 + MROI && (7) \\ \Longrightarrow r &= MROI. && (8) \end{aligned}$$

- That is, if not a corner solution, optimal investment is characterized by marginal return on investment equal to the real interest rate in the financial market.
- The intuition can be gleaned from Figure 5: If $MROI < r$, then $w = PV_r(Q)$ increases (i.e., the budget line thru Q shifts out) if you slightly **reduce** investment x, and if $MROI > r$, then w increases if you slightly **increase** investment x.

VIII. Fisher separation theorem

A. You might think at first that the optimal production point Q depends on the agent's preferences.

- That would be true if there were no financial market ("autarky"), but
- a financial market allows the agent to reshape her consumption stream as she pleases.

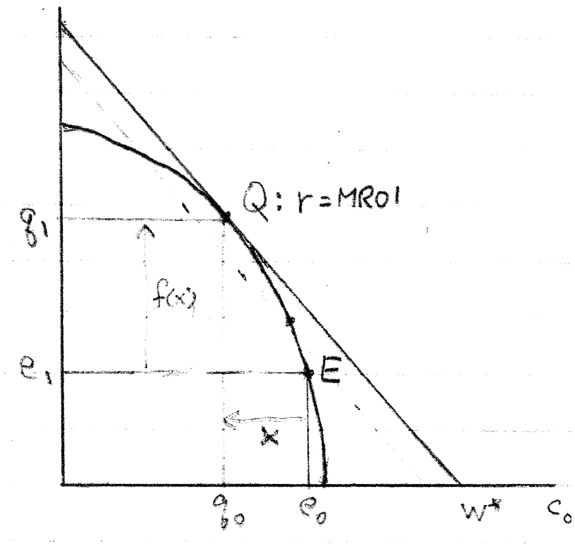


Figure 5: The optimal investment point Q is chosen along the PPF to maximize the budget line intercept w^*.

- By increasing w, she can consume more at every date.
- Thus as long as her preferences are monotone, she will maximize utility by maximizing $w = PV_r(Q)$.

B. Irving Fisher was the first to point this out clearly: optimal investment and production DOES NOT depend on preferences, as long as they are monotone. Given access to a financial market, optimal investment depends ONLY ON endowment and productive opportunities.

C. This "separation" result now seems trivial to obtain, but it still has far-reaching implications.

D. Suppose, for example, several different people own shares of a productive opportunity, and some owners are very patient (small MRTP) while others are impatient (large MRTP).

- According to the Fisher separation result, there is no need to quarrel.
- They should invest up to the point where MROI=r to maximize the present value w and thus maximize each of their shares $\alpha_i w$.

- Then each of them can borrow or lend their wealth (including that $\alpha_i w$) as they prefer.
- This is a crucial reason why corporations can exist, and why their primary goal is to maximize shareholder value.

E. editorial comments, not relevant to intertemporal equilibrium theory:

- In my opinion, a primary goal of government is to make and enforce laws that harmonize the public interest with maximizing shareholder value.
- Sometimes firms profess other goals such as serving customers, workers or the environment. These other goals may (and should) be consistent with maximizing shareholder value in the long run.

IX. Optimal individual borrowing, lending and consumption

A. Assuming that the agent has indeed maximized w, how does she choose her consumption stream? That is, how does she borrow and lend in the financial market to achieve optimal consumption over time?

B. More precisely, given the real interest rate r and her chosen production stream Q, how does she maximize utility $U(c_0, c_1)$?

C. Write her problem as

$$\max_{c_0, c_1 \geq 0} U(c_0, c_1) \text{ s.t. } PV_r(C) = PV_r(Q) = w \tag{9}$$

D. In terms of the amount $b = c_0 - q_0$ to borrow in the financial market (see Figure 6), her problem can be rewitten as just:

$$\max_b U(q_0 + b, q_1 - (1 + r)b) \tag{10}$$

E. The FOC for problem (10) (which is necessary and sufficient given our strong

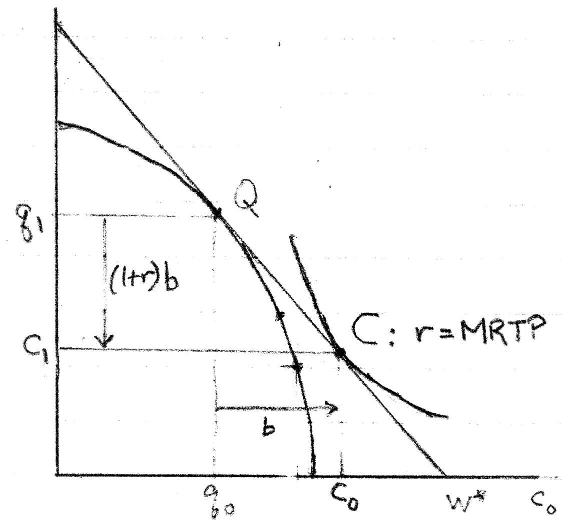


Figure 6: The optimal investment point Q is chosen along the PPF to maximize the budget line intercept w^*.

classical assumptions on U) is:

$$
\begin{aligned}
0 &= \partial_0 U \cdot \frac{d(q_0+b)}{db} + \partial_1 U \cdot \frac{d(q_1-(1+r)b)}{db} = \partial_0 U - \partial_1 U \cdot (1+r) \\
\implies (1+r) &= \frac{\partial_0 U}{\partial_1 U} = MRS = 1 + MRTP \\
\implies r &= MRTP. \qquad (11)
\end{aligned}
$$

F. That is, optimal consumption is characterized by marginal rate of time preference equal to the real interest rate in the financial market.

G. The intuition can be gleaned from Figure 6: If $MRTP < r$ at some point along the budget line, then utility increases if you slightly **reduce** borrowing b, and if $MRTP > r$, then utility increases if you slightly **increase** b.

H. Borrowing is positive in Figure 6, but it is easy to imagine (with steeper ICs, corresponding to greater impatience) that the tangency of the IC to the budget line could occur at a point C above Q instead of below, i.e., we could have $b < 0$.

I. In this case, $-b = \ell > 0$ is called lending.

X. Equilibrium real interest rate

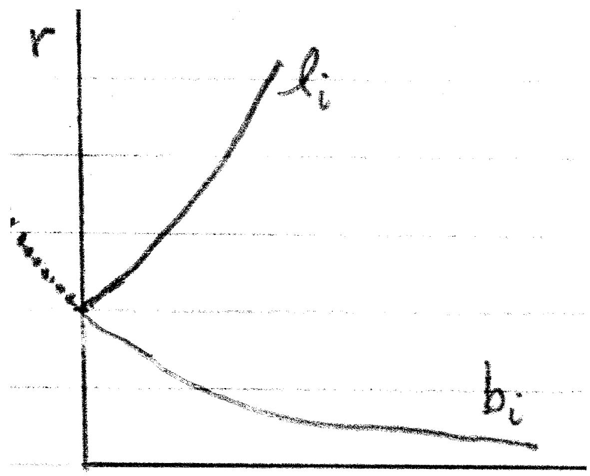


Figure 7: Amount borrowed ($b \geq 0$) or lent ($\ell \geq 0$) by agent i as functions of the real interest rate r.

A. Consider the impact of a change in the real interest rate r on optimal production Q and consumption C, and hence on borrowing b or lending ℓ.

B. An increase in r, i.e., a steeper budget line, will rotate the tangency point Q on the PPF clockwise, i.e., will increase q_0 and thereby tend to reduce b. This production effect follows from the concavity of the production function ($f'' < 0$).

C. The tangency point C on the indifference curve will, by the substitution effect, also rotate clockwise, i.e., will decrease c_0 and thereby also tend to reduce b. This follows from convex preferences.

D. There is also an income effect, since the new budget line will intersect a different indifference curve. If c_0 and c_1 are both normal goods, then this income effect will again reduce b for borrowers, but will decrease ℓ for lenders.

E. This income effect is the reason why the curves drawn in Figure bl1 get steeper at higher r.

F. The Figure also shows that there is some interest rate at which $Q = C$ and so $b = \ell = 0$. I like to call this the Polonius interest rate (after the Shakespeare character who said "neither a borrower nor a lender be") but it could also be

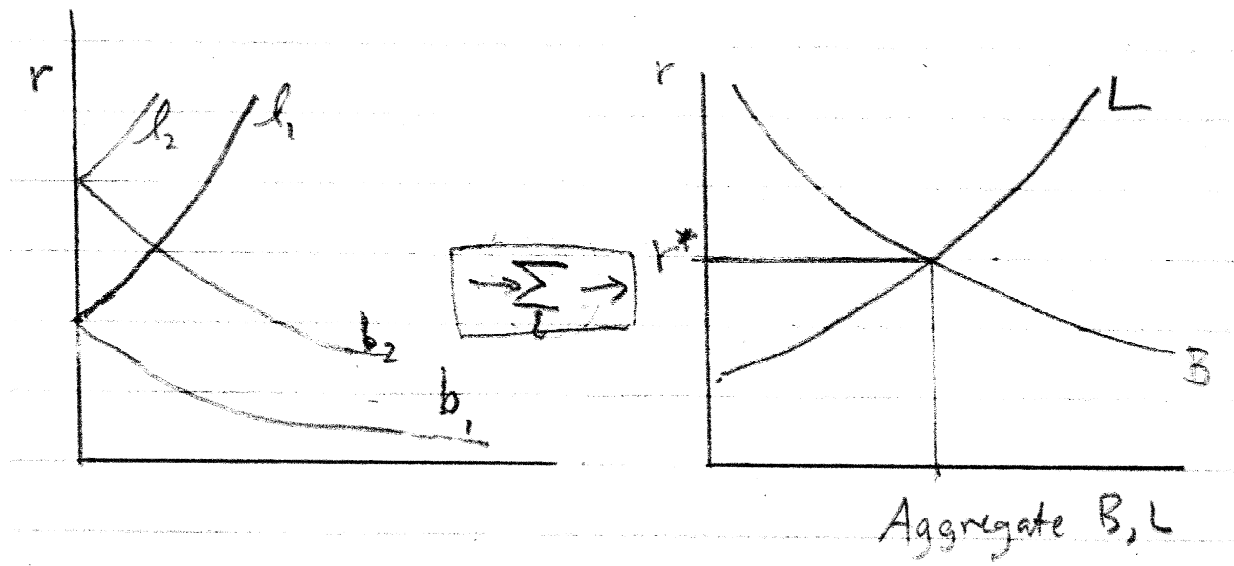


Figure 8: The equilibrium real interest rate r^* equates aggregate demand for loans to aggregate supply.

called the autarky interest rate.

G. Recall that there are $N > 1$ agents who participate in the financial market. Although we assume that they are all price-takers (i.e., each has negligible influence on r), their combined desires to borrow and lend determine the equilibrium real interest rate r^*.

H. Let $b_i(r) = \max\{0, b\}$ be agent i's borrowing curve as just derived, and $\ell_i(r) = \max\{0, \ell\}$ be her lending curve; they intersect the vertical axis and each other at her Polonius interest rate. The left panel of Figure 8 shows examples for $i = 1, 2$ where agent 1 has the lower Polonius interest rate, due to lower MROI and/or lower MRTP.

I. Aggregate borrowing is $B(r) = \sum_{i=1}^{n} b_i(r)$ and aggregate lending is $L(r) = \sum_{i=1}^{n} \ell_i(r)$. As long as agents have different Polonius r's, these curves will intersect at a positive value $B(r^*) = L(r^*)$ of financial activity.

J. Financial markets thus clear at a equilibrium real interest rate r^*. It is unique given our strong classical assumptions on production and utility functions.

K. Economists had long debated whether productivity or thrift was the main determinant of interest rates. Fisher showed how they work together to determine the (risk-free) equilibrium real interest rate.

   Ex A: The transcontinental railroad unites California's financial market with the eastern US. (Hint: combine B, L).

   Ex B: Info Tech increases productivity. (Hint: Use rep. agent, see impact on Polonius r)

   Ex C: Aging population in developed countries increases thrift.

XI. Extending the basic model: Money

A. Still assume just 2 dates and no uncertainty, but add a second good m called money.

   1. Money is the numeraire and is storable. Its definitive role as the medium of exchange will emerge in the next extension.
   2. The nominal financial market allows exchange at interest rate k, and the money price for consumption (i.e., the price level) in period $t = 0, 1$ is P_t.
   3. Using notation similar to that already introduced for the real financial market, we summarize all four competitive markets in the diagram

$$\begin{tikzcd} c_0 \arrow[r, "P_0"] \arrow[d, "1+r"'] & m_0 \arrow[d, "1+k"] \\ c_1 \arrow[r, "P_1"'] & m_1 \end{tikzcd}$$

   4. The bottom edge says that there is a competitive market in which you can sell each unit of c_1 for P_1 units of money, or buy $\frac{1}{P_1}$ units of c_1 for each unit of money. These forward market transactions are promises made at time $t = 0$ to be carried out at $t = 1$.
   5. The top edge, of course, is for transactions carried out now ($t = 0$) in the spot market, and the left and right edges are intertemporal transactions.

6. Write $\frac{P_1}{P_0} = 1 + \pi$, where π is called the **inflation rate**, the rate of change in the price level.

B. Notice that there are two ways to go from consumption (or corn) now to money later:

(a) along the top and left edges of the diagram (cashing out and lending), yielding $P_0 \cdot (1 + k)$ units of money at $t = 1$ per unit of c_0, or

(b) along the right and bottom edges of the diagram (real lending and cash out the repayment), yielding $(1 + r) \cdot P_1$ units of money at $t = 1$.

C. Arbitrage ensures that both ways yield the same amount of money at $t = 1$.

- If not, say $P_0 \cdot (1 + k) > (1 + r) \cdot P_1$.
- Then go around the rectangle clockwise, doing (b) backwards and (a) frontwards, i.e., cash out, lend money, buy c_1 (in the forward market) and borrow in the "real" financial market.
- For each unit of c_0 you start with, at the end of the round trip you have

$$P_0 \cdot (1 + k) \cdot \frac{1}{P_1} \cdot \frac{1}{(1 + r)} = \frac{P_0 \cdot (1 + k)}{(1 + r) \cdot P_1} > 1$$

  units of c_0.
- You (and everyone else) can keep doing this, accumulating more and more c_0 for free, until prices adjust.
- This arbitrage buying and selling will push down P_0 and k, and push up P_1 and r. Similarly, if prices are out of line the other way, the reverse pressures will be felt.
- Hence general (multimarket) equilibrium is only possibly when

$$P_0 \cdot (1 + k) = (1 + r) \cdot P_1. \tag{12}$$

D. Rewrite equation (12) as $(1 + k) = (1 + r)\frac{P_1}{P_0}$ and recall that the last factor is

$1 + \pi$. Expanding this expression, we get

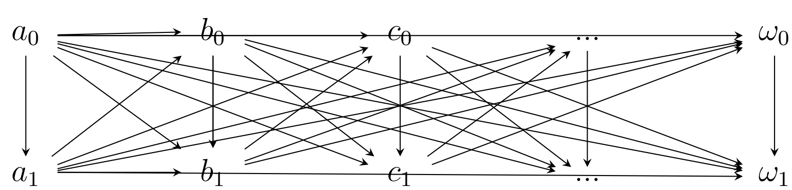
$$\begin{aligned} 1 + k &= (1 + r)(1 + \pi) \\ k &= r + \pi + r\pi \approx r + \pi \end{aligned} \tag{13}$$

E. Equation (13) is called **Fisher's equation**. It says that the (k)nominal interest rate is the sum of the real interest rate r and the inflation rate π...

- plus a cross-term which is very small if r and π are both moderate.
- The cross-term is 0 if you use continuously compounded interest and inflation rates.
- Here is a tangent (not needed for Econ 200, but possibly handy later) on continuous compounding (or growth rates).
    1. Over time interval $\Delta t > 0$, prices increase by factor $e^{\pi \Delta t}$ and real balances increase by factor $e^{r\Delta t}$.
    2. Hence nominal balances rise by factor $e^{k\Delta t} = e^{r\Delta t} e^{\pi \Delta t} = e^{(r+\pi)\Delta t}$.
    3. Taking logs and canceling the common factor $\Delta t > 0$, we see that with continuous compounding $k = r + \pi$.

XII. Extending the basic model: Many goods and transactions costs

A. The number of possible markets increases quadratically in the number of goods x dates. For example, with just two dates but goods $a, b, c, ...\omega$ a few of the possible markets are as in the diagram.

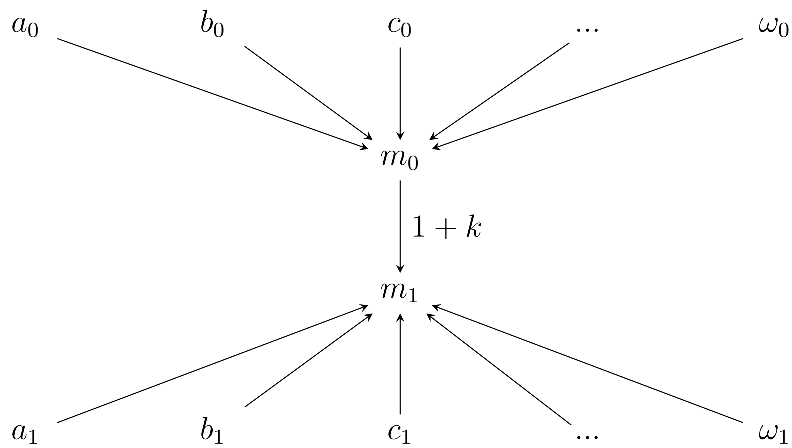


B. Even with just 1000 goods, there are $1000 \cdot 999/2 \approx 500,000$ different spot markets, plus the same number of forward markets, plus 1 million intertemporal markets, almost 2 million all together.

C. Even if the fixed and marginal costs are small for maintaining a market to trade one good against another, the logistics still get out of hand.

D. The universal solution is to abandon direct exchange ("barter") in favor of indirect exchange, using some particular good (called money) to mediate.

1. E.g., to trade a_0 for ω_0, first sell a_0 for m_0 and then use that to buy ω_0.
2. This only requires 1000 spot markets, instead of half a million.
3. Ditto for forward markets.
4. Even better: only **one** intertemporal market is needed, the nominal financial market discussed earlier.



E. Another way to put it: arbitrage renders most barter markets redundant, and they shrivel, saving transactions costs.

- In the example with 1000 goods and two dates, about 99.9% of transactions costs are saved; in realistic examples the savings are much more substantial.
- Even in a virtual economy designed to support barter, monetary exchange emerged spontaneously! [see Baumer and Kephart working paper].

F. To return to the main point, the "real" financial market does not exist in a multi-good world such as ours, only the nominal financial market. But the previous analysis lays bare the underlying forces of productivity and thrift.

XIII. Extending the basic model: Many periods

A. Suppose that money can be exchanged (borrowed and lent) for many periods T at a consistent per-period nominal interest rate k.

$$m_0 \xrightarrow{1+k} m_1 \xrightarrow{1+k} m_2 \xrightarrow{1+k} m_3 \xrightarrow{1+k} \ldots \xrightarrow{1+k} m_T$$

B. Then the $t = 1$ value of amount x of money held at date $t = 2$ is $\frac{x}{1+k}$ and its date $t = 0$ value (i.e., its present value) is $\frac{\frac{x}{1+k}}{1+k} = \frac{x}{(1+k)^2}$.

C. The diagram thus tells us that the present value of an amount x of money held at date t is $\frac{x}{(1+k)^t}$.

D. We conclude that the present value of an arbitrary money stream $X = (x_0, x_1, x_2, ..., x_T)$ is

$$PV_k(X) = \sum_{t=0}^{T} \frac{x_t}{(1+k)^t}. \tag{14}$$

- The formula applies even for an infinite horizon, $T = \infty$.
- In continuous time, the formula is

$$PV_k(X) = \int_{t=0}^{T} x_t e^{-kt} dt. \tag{15}$$

E. This brings us to the very important **intertemporal decision rule**:

If you have access to a financial market with nominal borrowing and lending rate k, then (no matter what your underlying preferences), you should strictly prefer cash stream $X = (x_0, x_1, x_2, ..., x_T)$ to cash stream $Y = (y_0, y_1, y_2, ..., y_T)$ if and only if $PV_k(X) > PV_k(Y)$.

- As before, the logic is that this choice makes your opportunity set as large as possible.
- What you choose to do with those financial market opportunities, by reshaping the cash stream to meet consumption needs or whatever else, of course does depend on your personal preferences.

F. What if the interest rate k differs from period to period?

- An arbitrage diagram similar to that above establishes the connections between one period rates for spot and forward trades (as above) and longer maturity spot rates.
- The graph of annualized spot rates k_t at all maturities is called the yield curve.
- The intertemporal decision rule still holds, except that the interest rates k_t in the PV formula come from the yield curve.

G. One last point about present value: it is a linear operator, i.e.,
$PV_k(X+Y) = PV_k(X) + PV_k(Y)$.
Financial economists refer to this property as value additivity, and use it to slice and dice financial assets.

XIV. A general formula for interest rates and asset yields:

$$k_a = r^* + \pi^e + RP_a \pm T_a \tag{16}$$

- Equation (16) says that any asset a has a yield k_a with four components.
- The first two components are the real interest rate r^* and the anticipated inflation rate π^e. By Fisher's equation, these are the only components of the risk-free nominal interest rate.
- Each asset a also has its own risk premium RP_a. Your finance course will feature theories (and evidence) on how the risk premium is determined in worlds where promises can be broken and cash streams are uncertain.
- There are also transactions costs T_a specific to asset a. These include tax; e.g., yields are higher on corporate bonds in the US than municipal bonds largely because corporate bonds have a higher tax rate. Use $+T_a$ for borrowing and $-T_a$ for lending.
- Equation (16) implies that something that increases inflation expectations (say) by 1% will push up **all** interest rates and asset yields by **exactly** 1%, assuming

that that something doesn't have a separate impact on the other components. Likewise, something (like an aging workforce) that decreases the real interest rate by 2% will depress all yields by that same amount.

XV. Basic asset pricing formula:

$$P_a = PV_k(Y - C) = PV_k(Y) - PV_k(C) \tag{17}$$

- Equation (17) says that the price of any asset a is determined by its associated revenue stream $Y = (y_0, y_1, y_2, ..., y_T)$ and cost stream $C = (c_0, c_1, c_2, ..., c_T)$.
- If a market exists for the asset, then rational investors will bid up the price P_a whenever it falls below $PV_k(Y - C)$, according to the intertemporal decision rule. Likewise, they will sell off any asset with price above the **fundamental value** $PV_k(Y - C)$.
- Many financial assets do not have a competitive market, or (in some cases) any market at all. In that case, equation (17) can be used to impute a value. This is what armies of financial analysts do all day long.
- Note that P_a depends on the interest rate $k = k_a$ used in the PV formula. For most assets, the revenue and cost streams are such that an increase in k decreases P_a. That is, asset prices and yields tend to move in opposite directions.