Problem Ch2.1

Solution: Assuming $v < c$ leads convergence to the interior steady state $s_H^* = \frac{v}{c}$. Let $v = 4, c = 5.5$, we have the payoff matrix $W_{HH} = -0.75, W_{HD} = 4, W_{DH} = 0, W_{DD} = 2$. The initial share of hawk is set to $s_H = 0.84$. Results can be found in Figure 1.

Hawk-Dove Game

V	4	0.727273
C	5.5	
Initial H share	0.84	

		Player 2	
		Hawk	Dove
Player 1	Hawk	(V-C)/2 = -0.75	V = 4
	Dove	0 = 0	V/2 = 2

t	s_H	s_D	W_H	W_D	Wbar
0	0.84000	0.16000	0.01000	0.32000	0.05960
1	0.14094	0.85906	3.33054	1.71812	1.94537
2	0.24129	0.75871	2.85386	1.51741	1.83989
3	0.37427	0.62573	2.22222	1.25146	1.61479
4	0.51506	0.48494	1.55347	0.96988	1.27046
5	0.62979	0.37021	1.00849	0.74042	0.90924
6	0.69853	0.30147	0.68197	0.60293	0.65814
7	0.72382	0.27618	0.56185	0.55236	0.55923
8	0.72721	0.27279	0.54573	0.54557	0.54569
9	0.72727	0.27273	0.54545	0.54545	0.54545
10	0.72727	0.27273	0.54545	0.54545	0.54545
11	0.72727	0.27273	0.54545	0.54545	0.54545
12	0.72727	0.27273	0.54545	0.54545	0.54545
13	0.72727	0.27273	0.54545	0.54545	0.54545
14	0.72727	0.27273	0.54545	0.54545	0.54545
15	0.72727	0.27273	0.54545	0.54545	0.54545
16	0.72727	0.27273	0.54545	0.54545	0.54545
17	0.72727	0.27273	0.54545	0.54545	0.54545
18	0.72727	0.27273	0.54545	0.54545	0.54545
19	0.72727	0.27273	0.54545	0.54545	0.54545
20	0.72727	0.27273	0.54545	0.54545	0.54545
21	0.72727	0.27273	0.54545	0.54545	0.54545
22	0.72727	0.27273	0.54545	0.54545	0.54545
23	0.72727	0.27273	0.54545	0.54545	0.54545
24	0.72727	0.27273	0.54545	0.54545	0.54545
25	0.72727	0.27273	0.54545	0.54545	0.54545
26	0.72727	0.27273	0.54545	0.54545	0.54545
27	0.72727	0.27273	0.54545	0.54545	0.54545
28	0.72727	0.27273	0.54545	0.54545	0.54545
29	0.72727	0.27273	0.54545	0.54545	0.54545
30	0.72727	0.27273	0.54545	0.54545	0.54545

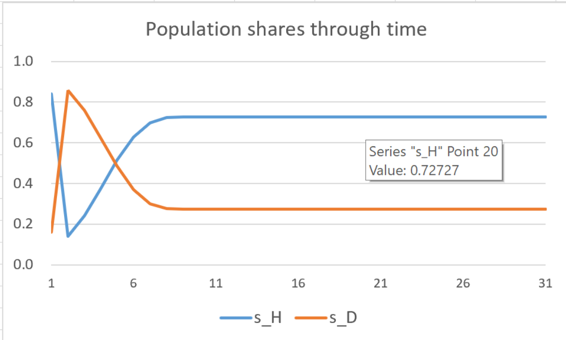

Figure 1: Evolution of Shares in HD Games

Problem Ch2.2 Following the tables shown in Section 2.7, three cases of RPS can be drawn. See Figure 2–Figure 4.

Problem Ch2.6 Let

$$W := \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix} = \begin{pmatrix} 1.00 & 1.18 & 0.88 \\ 0.85 & 1.00 & 1.16 \\ 1.13 & 0.86 & 1.00 \end{pmatrix}.$$

- Check Strategy-1 versus Strategy-2:

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} 1.00 & 1.18 \\ 0.85 & 1.00 \end{pmatrix}$$

  We have $w_1 = W_{12} - W_{22} > 0, w_2 = W_{21} - W_{11} < 0$. Thus Strategy-1 dominates Strategy-2.

- Check Strategy-2 versus Strategy-3:

$$\begin{pmatrix} W_{22} & W_{23} \\ W_{32} & W_{33} \end{pmatrix} = \begin{pmatrix} 1.00 & 1.16 \\ 0.86 & 1.00 \end{pmatrix}$$

  We have $w_1 = W_{23} - W_{33} > 0, w_2 = W_{32} - W_{22} < 0$. Thus Strategy-2 dominates Strategy-3.

- Check Strategy-3 versus Strategy-1:

$$\begin{pmatrix} W_{11} & W_{13} \\ W_{31} & W_{33} \end{pmatrix} = \begin{pmatrix} 1.00 & 0.88 \\ 1.13 & 1.00 \end{pmatrix}$$

  We have $w_1 = W_{13} - W_{33} < 0, w_2 = W_{31} - W_{11} > 0$. Thus Strategy-3 dominates Strategy-1.

	Matrix	R	P	S			
	R	1	0.5	1.5			
	P	1.5	1	0.5			
	S	0.5	1.5	1			
t	s_R	s_P	s_S	W_R	W_P	W_S	Wbar
0	0.32	0.33	0.35	1.01	0.985	1.005	1
1	0.3232	0.32505	0.35175	1.01335	0.985725	1.000925	1
2	0.32751472	0.320409911	0.352075369	1.015832729	0.987719676	0.996447596	1
3	0.332700172	0.316475174	0.350824655	1.017174741	0.990937759	0.991887501	1
4	0.338414211	0.313607199	0.34797859	1.017185695	0.99521781	0.987596494	1
5	0.344230094	0.31210747	0.343662436	1.015777483	1.000283829	0.983938688	1
6	0.349661179	0.312196055	0.338142766	1.012973355	1.005759206	0.981267438	1
7	0.354197457	0.313994057	0.331808486	1.008907214	1.011194486	0.9798983	1
8	0.35735237	0.317509059	0.325138571	1.003814756	1.0161069	0.980078344	1
9	0.358715582	0.322623145	0.318661272	0.998019063	1.020027155	0.981953782	1
10	0.358004989	0.329084369	0.312910641	0.991913136	1.022547174	0.98553969	1
11	0.355109852	0.336504292	0.308385856	0.985940782	1.023361998	0.99069722	1
12	0.350117285	0.344365704	0.305517011	0.980575653	1.022300137	0.99712421	1
13	0.343316486	0.352045107	0.304638408	0.976296651	1.019339039	1.004364311	1
14	0.335178735	0.358853321	0.305967944	0.973557312	1.014605395	1.011837293	1
15	0.326315708	0.364094515	0.309589777	0.972747631	1.008362966	1.018889404	1
16	0.317422832	0.367139425	0.315437743	0.974149159	1.000992545	1.024858297	1
17	0.309217185	0.367503827	0.323278988	0.97788758	0.992969098	1.029143321	1
18	0.302379645	0.364919944	0.332700411	0.983890234	0.984839617	1.03127015	1
19	0.297508379	0.359387618	0.343104003	0.991858192	0.977202188	1.030939619	1
20	0.295086123	0.351194367	0.35371951	1.001262572	0.970683306	1.028054122	1
21	0.295458691	0.340898509	0.3636428	1.011372146	0.965907945	1.022719909	1
22	0.29881869	0.329276578	0.371904732	1.021314077	0.963456979	1.015228944	1
23	0.305187734	0.317243817	0.377568448	1.030162315	0.963809643	1.006028042	1
24	0.314392903	0.30576265	0.379844447	1.037040898	0.967274228	0.995684874	1
25	0.326038299	0.295756332	0.37820537	1.041224519	0.973916464	0.984859017	1
26	0.339479071	0.288041961	0.372478969	1.042218504	0.983500051	0.974281445	1
27	0.353811369	0.283289283	0.362899348	1.039805032	0.995456011	0.964738957	1
28	0.367894842	0.28200202	0.350103138	1.034050559	1.008895852	0.957053589	1
29	0.380421867	0.284510668	0.335067465	1.025278399	1.022677201	0.9520444	1
30	0.390038323	0.290962573	0.318999104	1.014018265	1.035519609	0.950462125	1
31	0.395505983	0.30129745	0.303196566	1.000949558	1.046154709	0.952895734	1
32	0.395881539	0.315203746	0.288914714	0.986855484	1.053483412	0.959661104	1
33	0.390677868	0.332061918	0.277260214	0.972599148	1.056708827	0.970692025	1
34	0.379972961	0.35089276	0.269134278	0.959120759	1.055419342	0.9854599	1
35	0.364439955	0.370339006	0.265221039	0.947441016	1.049609458	1.002949526	1
36	0.345285361	0.388711324	0.266003315	0.938645996	1.039641023	1.021712981	1
37	0.324100722	0.404120238	0.27177904	0.933829401	1.026160841	1.040009758	1
38	0.302654783	0.414692363	0.282652854	0.933980245	1.010000965	1.05601879	1

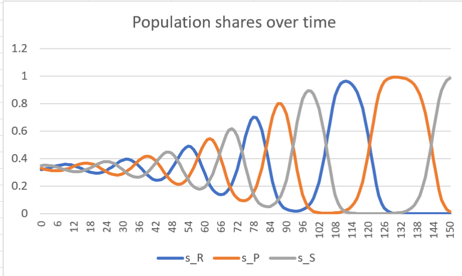

Figure 2: RPS: Spiral outward until edges

	Matrix	R	P	S			
	R	1	0.5	4			
	P	4	1	0.5			
	S	0.5	4	1			
t	s_R	s_P	s_S	W_R	W_P	W_S	Wbar
0	0.1	0.2	0.7	3	0.95	1.55	1.575
1	0.19047619	0.120634921	0.688888889	3.006349206	1.226984127	1.266666667	1.593247669
2	0.359415523	0.092902777	0.5476817	2.596593712	1.804405718	1.099000569	1.702792889
3	0.548073751	0.098446677	0.353479571	2.011215375	2.467481469	1.021303156	1.706219509
4	0.646044867	0.142370516	0.211584617	1.563568594	2.832342292	1.104089114	1.64698577
5	0.613323735	0.24483638	0.141839885	1.303101466	2.769051263	1.427847271	1.679713238
6	0.475809228	0.403619184	0.120571588	1.159905173	2.367141889	1.972952938	1.745199532
7	0.316235235	0.547458248	0.136306517	1.135190427	1.880552448	2.484257125	1.727131595
8	0.207851685	0.596088885	0.19605943	1.290133847	1.525525339	2.684340814	1.703795522
9	0.157387721	0.533719385	0.308892894	1.659818989	1.317716715	2.522464296	1.743697379
10	0.14981678	0.403333149	0.44685007	2.138883636	1.226025306	2.135091058	1.769002897
11	0.181141965	0.279534109	0.539323927	2.478204727	1.27376393	1.748031343	1.747722466
12	0.256852493	0.203728264	0.539419242	2.516393595	1.500847858	1.482758547	1.75193559
13	0.368930212	0.174529892	0.456539896	2.282354742	1.878520688	1.33912457	1.781251424
14	0.472718005	0.184060492	0.343221504	1.937634265	2.246543262	1.315822473	1.781073029
15	0.514271222	0.232163337	0.253565441	1.644614654	2.416030947	1.439354399	1.771662328
16	0.47739232	0.316603112	0.206004568	1.459712148	2.329174675	1.711113176	1.78677645
17	0.390007026	0.412711926	0.197281048	1.385487181	2.071380553	2.043132266	1.798304467
18	0.300477335	0.475383047	0.224139617	1.434727328	1.789362197	2.275910475	1.791857201
19	0.240590067	0.474721118	0.284688815	1.616705885	1.579425794	2.303868321	1.794635698
20	0.216736677	0.417793305	0.365470018	1.887513403	1.467475021	2.145011576	1.806132041
21	0.226502478	0.33945538	0.434042142	2.132398736	1.462486362	1.905114902	1.806342614
22	0.267387589	0.27483649	0.45777592	2.235909516	1.573274808	1.690815676	1.804262484
23	0.331356696	0.23965101	0.428992294	2.167151378	1.779573941	1.553274681	1.810919681
24	0.396538912	0.23550282	0.367958268	1.986123393	2.005637602	1.508239004	1.814877533
25	0.433955016	0.2602563	0.305788684	1.787237901	2.148970708	1.563791392	1.813053729
26	0.427775989	0.308475781	0.26374823	1.637006798	2.151453853	1.711539349	1.815359084
27	0.385748588	0.365586849	0.248664563	1.563200264	2.032913482	1.903886254	1.819637772
28	0.331385896	0.408436474	0.26017763	1.576314653	1.864068874	2.059616473	1.819588294
29	0.28708057	0.41842087	0.29449856	1.674285245	1.713992429	2.111722325	1.81972415
30	0.264136057	0.394109296	0.341754647	1.828209293	1.621530848	2.050259859	1.822642209

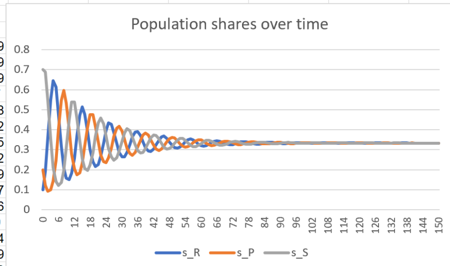

Figure 3: RPS: Spiral in to the center

	Matrix	R	P	S			
	R	1	0.5	2			
	P	2	1	0.5			
	S	0.5	2	1			
t	s_R	s_P	s_S	W_R	W_P	W_S	Wbar
0	0.1	0.2	0.7	1.6	0.75	1.15	1.115
1	0.143497758	0.134529148	0.721973094	1.65470852	0.782511211	1.062780269	1.110016288
2	0.213913043	0.094836957	0.69125	1.643831522	0.868288043	0.987880435	1.11685515
3	0.314845666	0.073730058	0.611424275	1.574559246	1.009133529	0.916307225	1.13039911
4	0.438555861	0.065820535	0.495623604	1.462713337	1.190744059	0.846542604	1.139423414
5	0.562987822	0.06878515	0.368227027	1.333834452	1.378874309	0.787291239	1.135680543
6	0.66121812	0.08351475	0.25526713	1.213509755	1.533584555	0.75290569	1.122663644
7	0.71472399	0.114083084	0.171192926	1.114151384	1.629127527	0.756721089	1.111711913
8	0.716292336	0.167179905	0.116527759	1.032937806	1.658028456	0.809033737	1.111349363
9	0.665754135	0.249416655	0.08482921	0.960120882	1.62333953	0.916539587	1.121841692
10	0.569781327	0.360913592	0.069305081	0.888848284	1.535128787	1.076022929	1.135071856
11	0.446182462	0.48811786	0.065699678	0.821640748	1.413332622	1.265026629	1.139586429
12	0.321697137	0.60537128	0.072931583	0.770245943	1.285231345	1.444522711	1.131179387
13	0.219050946	0.687814995	0.09313406	0.749226562	1.172483916	1.578289522	1.117563316
14	0.146854129	0.721616401	0.13152947	0.770721269	1.081089394	1.648189337	1.110100908
15	0.101957939	0.702757589	0.195284472	0.843905677	1.004315703	1.65177862	1.114400081
16	0.077210048	0.633336711	0.289453241	0.972784886	0.932483427	1.594731687	1.12728501
17	0.066628019	0.523892345	0.409479636	1.147533464	0.861888201	1.490578335	1.138355986
18	0.067165177	0.396656789	0.536178033	1.337849639	0.799076161	1.3630742	1.137666337
19	0.078983535	0.278604521	0.642411945	1.503109685	0.757777562	1.239112753	1.125862004
20	0.105448905	0.187518767	0.707032328	1.613272945	0.751932741	1.134794314	1.113455632
21	0.152783695	0.126634143	0.720582162	1.65726509	0.792492614	1.050242296	1.110345571
22	0.228039892	0.090383234	0.681576874	1.636385257	0.887251455	0.976363288	1.118820411
23	0.333530845	0.071676075	0.59479308	1.558955042	1.036134305	0.904910652	1.132460227
24	0.459141593	0.065579381	0.475279026	1.442489336	1.22150208	0.836008584	1.139749547
25	0.581098587	0.070283292	0.348618121	1.313476475	1.406789527	0.779733998	1.133962524
26	0.673090431	0.08719318	0.239716389	1.196119799	1.553232237	0.750647964	1.120470668
27	0.718534464	0.120869972	0.160595564	1.100160578	1.638236682	0.76160274	1.110826935
28	0.711634969	0.17825785	0.11010718	1.020978255	1.656581379	0.822440366	1.112419055
29	0.653138605	0.26545629	0.081405105	0.94867696	1.612436052	0.938886988	1.124079032
30	0.551222403	0.380783985	0.067993612	0.87760162	1.517225597	1.105172783	1.136633572

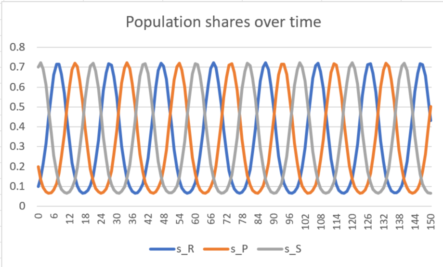

Figure 4: RPS: Get cycling forever with the same amplitude

From the above analysis, we know Strategy-1 dominates Strategy-2, Strategy-2 dominates Strategy-3, and Strategy-3 dominates Strategy-1, which shows it is a *true-RPS*. The dynamics is shown in Figure 5. It shows the tendency of spiral into the center.

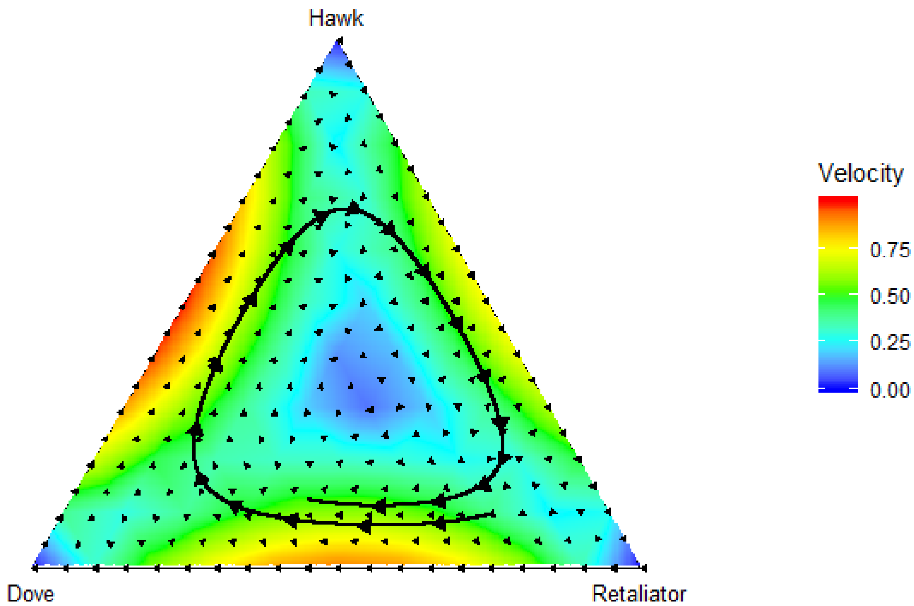

Figure 5: RPS Dynamics

Extra Credits:

Problem Ch2.3

Solution:

a.

$$P = \begin{pmatrix} 0 & 5 \\ -1 & 3 \end{pmatrix}$$

Since $w_1 = 5-3 = 2 > 0, w_2 = (-1) - 0 = -1 < 0$, we know the strategy-A dominates strategy-B.

b. Note $s_B = 1-x$.

$$\overline{W}(x) = s_A W_A + s_B W_B = s_A(5s_B) + s_B(-s_A + 3s_B) = 5x(1-x) + (1-x)[-x+3(1-x)] = -x^2 - 2x + 3.$$

c.

$$\frac{d\overline{W}(x)}{dx} = -2 - 2x.$$

The payoff is decreasing since $\frac{d\overline{W}(x)}{dx} = -2-2x < 0$ when $x \in [0,1]$. According to Figure 2.3 in the textbook, we know it is a PD since strategy-A is dominant and $\overline{W}(x)$ is decreasing to zero when $s_A = 1$.

d.

$$P = \begin{pmatrix} 3 & 5 \\ -1 & 0 \end{pmatrix}$$

Since $w_1 = 5-0 = 5 > 0, w_2 = (-1) - 3 = -4 < 0$, we know the strategy-A dominates strategy-B. Note $s_B = 1-x$.

$$\overline{W}(x) = s_A W_A + s_B W_B = s_A(3s_A + 5s_B) + s_B(-s_A) = 5x(1-x) + (1-x)[-x+3(1-x)] = 4x - x^2.$$

$$\frac{d\overline{W}(x)}{dx} = 4 - 2x.$$

The payoff is increasing since $\frac{d\overline{W}(x)}{dx} = 4-2x > 0$ when $x \in [0,1]$. According to Figure 2.3 in the textbook, we know it is a LF since strategy-A is dominant and $\overline{W}(x)$ is increasing from zero (when $s_A = 0$).

Problem Ch2.4

Solution:

$$\bar{w}(x) = s_A w_A(x) + s_B w_B(x) = s_A(s_A w_{AA} + s_B w_{AB}) + s_B(s_A w_{BA} + s_B w_{BB}).$$

Note $s_B = 1-x$. We obtain

$$\bar{w}(x) = (w_{AA} - w_{AB} - w_{BA} + w_{BB})x^2 + (w_{AB} + w_{BA} - 2w_{BB})x + w_{BB}.$$

Problem Ch2.5

Solution:

(a)

$$\bar{w}'(x) = \frac{d\bar{w}(x)}{dx} = 2(w_{AA} - w_{AB} - w_{BA} + w_{BB})x + w_{AB} + w_{BA} - 2w_{BB}$$

To ensure x is increasing, firstly, we need $\bar{w}'(x) \geq 0, \forall x \in [0,1] \Leftrightarrow \bar{w}'(0) \geq 0, \bar{w}'(1) \geq 0$, i.e.,

$$w_{AB} + w_{BA} - 2w_{BB} \geq 0, \text{and } 2w_{AA} - w_{AB} - w_{BA} \geq 0.$$

Noting $\bar{w}'(x) = 0, \forall x \in [0,1]$ cannot ensure a strictly increasing over x, we further need $(w_{AB} + w_{BA} - 2w_{BB})(2w_{AA} - w_{AB} - w_{BA}) \neq 0$. Thus the condition is:

$$w_{AB} + w_{BA} - 2w_{BB} \geq 0, \;\; 2w_{AA} - w_{AB} - w_{BA} \geq 0, \text{ and } (w_{AB} + w_{BA} - 2w_{BB})(2w_{AA} - w_{AB} - w_{BA}) \neq 0.$$

Similarly, the condition for decreasing in x should be

$$w_{AB} + w_{BA} - 2w_{BB} \leq 0, \;\; 2w_{AA} - w_{AB} - w_{BA} \leq 0, \text{ and } (w_{AB} + w_{BA} - 2w_{BB})(2w_{AA} - w_{AB} - w_{BA}) \neq 0.$$

(b) In this problem, essential condition (a) $w_{DC} > w_{CC} > w_{DD} > w_{CD}$ corresponds to

$$w_{BA} > w_{AA} > w_{BB} > w_{AB}.$$

Recall that

$$w_A(x) = s_A w_{AA} + s_B w_{AB} = x(w_{AA} - w_{AB}) + w_{AB},$$

$$w_B(x) = s_A w_{BA} + s_B w_{BB} = x(w_{BA} - w_{BB}) + w_{BB}.$$

The difference is

$$w_B(x) - w_A(x) = x(w_{BA} - w_{BB} - w_{AA} + w_{AB}) + w_{BB} - w_{AB},$$

which satisfies $w_B(x) - w_A(x) > 0, \forall x \in [0, 1]$ when essential condition (a) holds. Verified!

Essential condition (b) $w_{CC} > \frac{1}{2}(w_{DC} + w_{CD}) > w_{DD}$ corresponds to

$$w_{AA} > \frac{1}{2}(w_{BA} + w_{AB}) > w_{BB}.$$

Recall in Ch2.5(a) we have

$$\bar{w}'(x) = \frac{d\bar{w}(x)}{dx} = 2(w_{AA} - w_{AB} - w_{BA} + w_{BB})x + w_{AB} + w_{BA} - 2w_{BB}.$$

Let $y = 1 - x$, we obtain

$$\bar{w}'(y) = \frac{d\bar{w}(y)}{dy} = -\frac{d\bar{w}(x)}{dx} = 2(w_{AA} - w_{AB} - w_{BA} + w_{BB})y - 2w_{AA} + w_{AB} + w_{BA}.$$

Check $\bar{w}'(y = 0)$ and $\bar{w}'(y = 1)$, we have

$$\bar{w}'(y = 0) = -2w_{AA} + w_{AB} + w_{BA} < 0,\ \bar{w}'(y = 1) = 2w_{BB} - w_{AB} - w_{BA} < 0.$$

From Ch2.5(a) we know, this implies that the mean payoff is a decreasing function of $y = 1 - x$. Verified!

This is sufficient for a social dilemma since condition (a) shows strategy-D is dominant, and condition (b) shows the mean payoff is decreasing when the game converges to strategy-D, i.e. $s_D = 1$.

Problem Ch2.7

Solution:

Recall that

$$w_A = s_A w_{AA} + s_B w_{AB},$$

$$w_B = s_A w_{BA} + s_B w_{BB}.$$

The difference is

$$w_A - w_B = s_A(w_{AA} - w_{BA}) + s_B(w_{AB} - w_{BB}) = -s_A w_2 + s_B w_1.$$

When $w_1 = w_2 = 0$, we have $w_A = w_B$, which means the fitnesses of both strategies are the same. It also indicates that when player-2 fixes her strategy, the change of player-1's strategy would not change player-1's fitness. Thus the game does not evolve and the initial state is steady.

Expression $w_1 > w_2 = 0$ indicates that when player-2 fixes her strategy to strategy-A, the change of player-1's strategy would not change player-1's fitness. It also indicates that when player-2 fixes her strategy to strategy-B, the change of player-1 should choose strategy-A. Thus the game will evolve to strategy-A.

Expression $w_1 < w_2 = 0$ indicates that when player-2 fixes her strategy to strategy-A, the change of player-1's strategy would not change player-1's fitness. It also indicates that when player-2 fixes her strategy to strategy-B, the change of player-1 should also choose strategy-B. Thus the game will evolve to strategy-B.