Problem Set 4 Answer Key

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Problem 1

(a) Assume that the procedure is common knowledge. Since B knows $x \in [0, 1000]$, if he sees $y \geq 1000$, it must be that $y = 2x$, so B won't trade. Knowing that B won't trade if $y \geq 1000$, A shouldn't trade if $x \geq 500$, since it this case B will only trade if $y = \frac{1}{2}x$, when it is disadvantageous to her. Thus, the best response for A is "no trade" if $x \geq 500$. With this in mind, If $500 < y < 1000$ and A is willing to trade, then B knows that $x < y$. So B won't trade in this case. Now, knowing that B won't trade if $y > 500$, A won't trade if $x \geq 250$. And so forth.
Using the logic of iterated dominance, A and B won't trade if $x > 0$ or $y > 0$. There is no equilibrium in which trade occurs for positive amounts in the envelopes. Trade can only occur in PBE if $x = y = 0$, an event with zero probability.

(b) This student compute post-trade values using prior probability, which is not correct. He should use Bayesian posteriors, taking into account the realized values seen by each trader and the logic of iterated dominance. As we saw, such posteriors always lead to "no trade" as the best response, especially for higher realized values.

Problem 2

(a)

$$WTP_L = -(-1 - \frac{1}{2} \cdot 0.2 \cdot 1) = 1.1$$
$$WTP_H = -(-2 - \frac{1}{2} \cdot 0.2 \cdot 16) = 3.6$$

(b)

$$P^* = 0.8 \cdot 1 + 0.2 \cdot 2 = 1.2$$

(c) In this case, $P^* = 1.6$. Then only high-risk students will find it worthwhile to join SI.

(d) $\Pi_{SI} = (1.6 - 2) \cdot 20000 \cdot 0.2 = -1600$. In this case, SI's profit is -1600.
SI can increase profits screens students using two different types of plans: the first plan with high deductible and low premium targeting the low-risk students, and the second one with low deductible and high premium targeting high-risk students. An upper bound on SI's profit in this case is $\Pi_{SI} = (3.6 - 2) \cdot 20000 \cdot 0.2 + (1.1 - 1) \cdot 20000 \cdot 0.8 = 8000$.

Problem 3

(a) For P's maximization problem,

$$\max_{x,s}(x - s), \text{ s.t. } s - \frac{1}{2}x^2 \geq 0$$

Plug $s = \frac{1}{2}x^2$ into $(x - s)$ and FOC w.r.t. x

$$\Longrightarrow x^* = 1, s^* = \frac{1}{2}$$

(b) For P's maximization problem,

$$\max_x (ax + b - \frac{1}{2}x^2)$$

FOC w.r.t. x, we have $x^* = a$, where $a \geq 0$ and $b - \frac{1}{2}a^2 \geq 0$ should be satisfied.
Plug $x^* = a$ into P's maximization problem,

$$\max_{x,s}(x - s), \text{ s.t. } a \geq 0 \text{ and } b - \frac{1}{2}a^2 \geq 0$$

$$\Longrightarrow a^* = 1, b^* = -\frac{1}{2}, x^* = 1, s^* = \frac{1}{2}$$

(c) No. According to (a), we know that the highest utility available for P is $s^* = \frac{1}{2}$ no matter what the wage schedule is. According to (b), with linear wage schedule $s(x) = ax + b$, the highest utility P can obtain is also $\frac{1}{2}$. Thus, P couldn't obtain higher utility using a non-linear wage schedule.

Problem 4

(a) In equilibrium, there should be $q_1^* = q_2^* = \frac{1}{2}Q$.
For firm 1,

$$\max_{q_1} \ P(Q)q_1 - cq_1$$

FOC w.r.t. $q_1 \Longrightarrow P'(Q)q_1 + P(Q) = c$
Plug $q_1^* = q_2^* = \frac{1}{2}Q$ into the first-order condition $\Longrightarrow -\frac{1}{2}Q + a - Q = c \Longrightarrow Q^* = \frac{2}{3}(a - c)$
Thus, $q_1^* = q_2^* = \frac{1}{2}Q^* = \frac{1}{3}(a - c)$, $\pi_1^* = \pi_2^* = \frac{1}{9}(a - c)^2$.

(b) In monopoly case,

$$\max_{Q_m} \ \pi_m = P \cdot Q_m - c \cdot Q_m = -Q_m^2 + (a - c)Q_m$$

FOC w.r.t. $Q_m \Longrightarrow -2Q_m + a - c = 0 \Longrightarrow Q_m^* = \frac{1}{2}(a - c), \pi_m^* = \frac{1}{4}(a - c)^2$.

(c) If they sustain the collusive equilibrium, each of them can get profit $\frac{1}{8}(a - c)^2$ in every period. Present value of total profits in infinite periods for each of them is,

$$PV = \sum_{t=0}^{\infty} d^t \cdot \frac{1}{8}(a - c)^2 = \frac{(a - c)^2}{8(1 - d)}$$

If firm 1 chooses to break this collusive equilibrium at $t = 0$, then from $t = 1$, there will be duopoly equilibrium which is same as part (a). From $t = 1$ to infinity, both firms will have $q_1^* = q_2^* = \frac{1}{2}Q^* = \frac{1}{3}(a - c)$, $\pi_1^* = \pi_2^* = \frac{1}{9}(a - c)^2$ at every period. At $t = 0$, firm 2 will still sustain collusive output, which means $q_2 = \frac{1}{4}(a - c)$ at $t = 0$. So firm 1 will choose his output at $t = 0$ to maximize his profit.
At $t = 0$, $\pi_1 = P(q_1 + \frac{a-c}{4})q_1 - cq_1 = (a - q_1 - \frac{a-c}{4})q_1 - cq_1$
FOC w.r.t. $q_1 \Longrightarrow a - q_1 - \frac{a-c}{4} - q_1 - c = 0 \Longrightarrow q_1^* = \frac{3}{8}(a - c), \pi_1^* = \frac{9}{64}(a - c)^2$.
Firm 1 should choose $q_1^* = \frac{3}{8}(a - c)$ at $t = 0$ to maximize his profit.
In this case, firm 1's present value of profits in infinite periods is,

$$PV' = \frac{9}{64}(a - c)^2 + \sum_{t=1}^{\infty} d^t \cdot \frac{1}{9}(a - c)^2 = \frac{9}{64}(a - c)^2 + \frac{d(a - c)^2}{9(1 - d)}$$

If $PV > PV'$, both firms will sustain the collusive equilibrium,

$$\frac{(a - c)^2}{8(1 - d)} > \frac{9}{64}(a - c)^2 + \frac{d(a - c)^2}{9(1 - d)} \Longrightarrow d > \frac{9}{17}$$

Thus, if $d \geq \frac{9}{17}$, there will be a collusive equilibrium.

Problem 5

(a) For firm i's maximization problem,

$$\max_{q_i} \ P(q_i + q_j) \cdot q_i - c_i q_i$$

FOC w.r.t. $q_i \implies P'(q_i + q_j) \cdot q_i + P(q_i + q_j) - c_i = 0$, so $-q_i + 100 - q_i - q_j - c_i = 0$
Therefore, $q_i^* = \frac{1}{2}(100 - q_j - c_i)$, where $100 - q_j - c_i \geq 0$ should be satisfied. Otherwise, $q_i^* = 0$.

(b) Using (a) we see that

$$q_1^* = \frac{1}{2}(100 - q_2 - 25) = \frac{1}{2}(75 - q_2), \text{ if } q_2 \leq 75 \tag{1}$$

$$q_2^* = \frac{1}{2}(100 - q_1 - 55) = \frac{1}{2}(45 - q_2), \text{ if } \ q_1 \leq 45 \tag{2}$$

From (1), (2) $\implies q_1^* = 35, \ q_2^* = 5, \ \pi_1^* = 1225, \ \pi_2^* = 25.$

(c) By equation (1), the smallest value of q_2 such that firm 1's best response is to produce 0 is 75.

$$\pi_2(q_2 = 75) = (100 - 75) \cdot 75 - 55 \cdot 75 = -2250 < \pi_2^* = 25$$

There is no NE with this payoff vector because π_2 is negative in this case, which can be avoided by choosing some other q_2. Also, when $q_1 = 0$, firm 2's best response is $q_2^* = 22.5 \neq 75$.

(d) Since we know firm 2's output will be affected by firm 1's output and firm 1 chooses his output first, we can plug q_2^* into firm 1's maximization problem.
If $q_1 > 45, \ q_2^* = 0$, firm 1's problem is,

$$\max_{q_i} \ (100 - q_1)q_1 - 25q_1 = -q_1^2 + 75q_1$$

$$q_1^* = 45, \ \pi_1^* = 1350$$

If $0 < q_1 \leq 45$, $q_2^* = \frac{1}{2}(45 - q_2)$, firm 1's problem is,

$$\max_{q_i} \ (100 - q_1 - \frac{45 - q_1}{2})q_1 - 25q_1$$

$$q_1^* = 45, \ q_2^* = 0, \ \pi_1^* = 1350$$

Thus, the Stackelberg equilibrium is $(q_1^* = 45, q_2^* = 0)$.

Problem 6 (extra credit)

The child maximizes his income, $V = I_C(A) + B$, choosing A and taking into account that his action affects the bequest. The first order condition is

$$\frac{dI_C(A)}{dA} + \frac{dB}{dA} = 0.$$

The parent maximizes $W(U, V) = W(I_P(A) - B, I_C(A) + B)$ choosing B. The first order condition is

$$-W_U(I_P(A) - B, I_C(A) + B) + W_V(I_P(A) - B, I_C(A) + B) = 0.$$

We can think B as a function of A. Differentiate the first order condition with respect to A.

$$-W_{UU} \cdot (\frac{dI_P(A)}{dA} - \frac{dB}{dA}) - W_{UV} \cdot (\frac{dI_C(A)}{dA} + \frac{dB}{dA})$$
$$+W_{VU} \cdot (\frac{dI_P(A)}{dA} - \frac{dB}{dA}) + W_{VV} \cdot (\frac{dI_C(A)}{dA} + \frac{dB}{dA}) = 0.$$
$$(W_{UU} - W_{VU}) \cdot (\frac{dI_P(A)}{dA} - \frac{dB}{dA}) = (W_{VV} - W_{UV}) \cdot (\frac{dI_C(A)}{dA} + \frac{dB}{dA})$$

By the child's first order condition, the equation above becomes

$$(W_{UU} - W_{VU}) \cdot (\frac{dI_P(A)}{dA} - \frac{dB}{dA}) = 0.$$
$$\therefore \frac{dI_P(A)}{dA} = \frac{dB}{dA}$$

Substitute this into the child's first order condition.

$$\frac{dI_C(A)}{dA} + \frac{dI_P(A)}{dA} = 0.$$

This implies that the child chooses the A that maximizes the family aggregate income, $I_C(A) + I_P(A)$.

The first-best outcome from the parent's perspective is to maximizes her objective function. The first order conditions for the first-best outcomes are, with respect to B and A respectively,

$$W_U = W_V,$$
$$W_U \cdot \frac{dI_P(A)}{dA} + W_V \cdot \frac{dI_C(A)}{dA} = 0.$$
$$\therefore \frac{dI_P(A)}{dA} + \frac{dI_C(A)}{dA} = 0.$$

Thus, the first-best outcome from the parent's perspective is to maximize the family's aggregate income.

Problem 13.B.3

(a) As we know, only workers with $r(\theta) \leq w$ will accept this wage. Since r is decreasing, then only workers with $\theta \geq \theta^*$ will accept this wage, which means the more capable workers are the ones who will work at any given wage.

(b) Assume the firm offer the wage $w = \overline{\theta}$, then $r(\overline{\theta}) > \overline{\theta}$. In this case, $r(\overline{\theta}) > w$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. So no worker will accept this wage. The competitive equilibrium is P.O..

(c) If $w < \hat{\theta}$, then there exists θ^*, such that $r(\theta^*) = w$ and only workers with $\theta \geq \theta^* > \hat{\theta}$ will accept this wage. In this case, $E(\theta|\theta \geq \hat{\theta}) > w$, the market isn't clear. If $w = \hat{\theta}$, only workers with $\theta \geq \hat{\theta}$ will accept this wage. Similarly, $E(\theta|\theta \geq \hat{\theta}) > w = \hat{\theta}$
Thus, C.E. can only be obtained when $w > \hat{\theta}$, which means some workers with $\theta > \hat{\theta}$ or $\theta < \hat{\theta}$ will accept this wage and there is over-employment in this case because only workers with $\theta \geq \hat{\theta}$ will work with perfect information.

Problem 13.C.4

For workers' maximization problem,

$$\max_e \; w(e) - c(e, \theta) = w(e) - \frac{e^2}{\theta}$$

FOC w.r.t. $e \Longrightarrow w'(e) = \frac{2e^*}{\theta}$ Since the firm is competitive, there should be $w(e) = \theta$ in C.E.. Therefore, $w'(e) = \frac{2e^*}{w(e)}$ and $w(e) = \sqrt{2}e$.
Thus, the unique PBE is $e = \frac{\sqrt{2}}{2}\theta$, $w(e) = \sqrt{2}e$.