University of California Santa Cruz
Econ/Cmps 166b / 272 - Evolutionary Game Theory
Profs Friedman & Sinervo – Winter 2017

Answer Key for Problem Set 3[1]

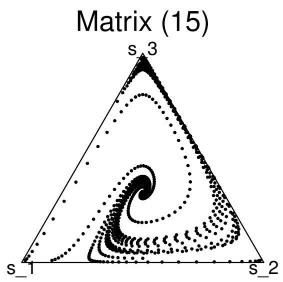
Question 1

The article Bomze (1983) is posted on the class website. Pick one of the 3x3 matrices listed in Figure 6 of that article, and sector its state space. Find and categorize all steady states (as source, center, sink or saddle point). Be sure to choose a different matrix than others in your group, and check each others' work. The chosen matrices all should have an interior steady state (so matrices 7-17 are eligible, but most others are not).

We adapted the code provided for us to calculate the below matrices:
For matrix 15, the values are,

$$M_{15} = \begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 3 & 0 & -1 \end{pmatrix}$$

Figure 1: Dynamics



We have the following steady states:

$$\begin{cases} (1,0,0) \rightarrow \textit{source} \\ (0,1,0) \rightarrow \textit{sink} \\ (0,0,1) \rightarrow \textit{sink} \\ (1/3,1/3,1/3) \rightarrow \textit{center} \end{cases}$$

[1]Thanks to Fernando Chertman, Swati Sharma, and William Sump !

Note from Instructor: This and several other matrix analyses that students submitted do establish, via simulation, the nature of most equilibrium points. But the analysis is often incomplete in that:

- The three Δw_{i-j} lines are not graphed.
- The directional arrows in the 6 resulting sectors are not included explicitly.
- The status of equilibria as NE and/or ESS is not mentioned.
- 'Center' is misconstrued. It refers to an equilibrium that is not a source, sink or saddle, but rather is one with whose nearby trajectories are closed loops around it.

A more complete analysis follows.

Let the three strategies be named by a, b and c. The fitness is given by

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

First, consider about the interior part where $s_i > 0$, $i = a, b, c$.

$$\Delta w_{a-b} = -s_a + s_b$$

$$\Delta w_{a-c} = -s_a + s_c$$

$$\Delta w_{b-c} = -s_b + s_c$$

The interior equilibrium is given by

$$\Delta w_{a-b} = \Delta w_{a-c} = 0,$$

together with $s_a + s_b + s_c = 1$. The solution is

$$s_a^* = s_b^* = s_c^* = \frac{1}{3}$$

Then, the triangle can be sectorized by $\Delta w_{a-b} = 0$, $\Delta w_{a-c} = 0$ and $\Delta w_{b-c} = 0$, which separate the space into sectors I to VI.

Every place in sector I satisfies $w_a > w_b$ and $w_b > w_c$, that is, $w_a > w_b > w_c$. As we know that the share of species with the highest fitness must increase and the share of species with the lowest fitness must decrease, the direction of dynamics in section should be towards point $s_a = 1$ and away from $s_c = 1$. Similarly, dynamics in section II to VI could be determined; by symmetry, they are reflections of section I. Figure 3 shows the dynamics in each section.

Next, consider the edges, starting with the a-b ($s_c = 0$) edge. The fitness is given by the first 2×2 matrix. The equilibrium is

$$s_a^* = s_b^* = \frac{1}{2}, \quad s_c = 0$$

which is a saddle point. By symmetry, the cases for the other two boundaries are the same.

Therefore, Figure 4 gives the whole dynamics.

<u>Question 2</u>

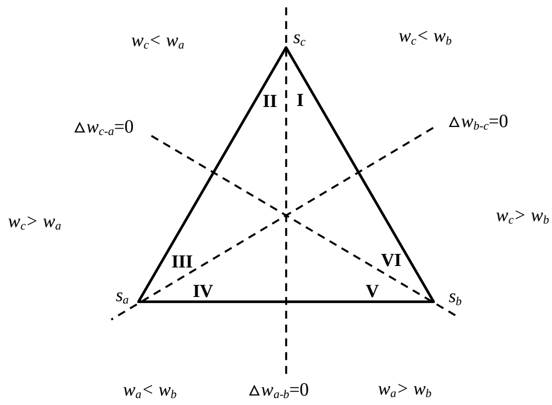


Figure 2: Sectoring

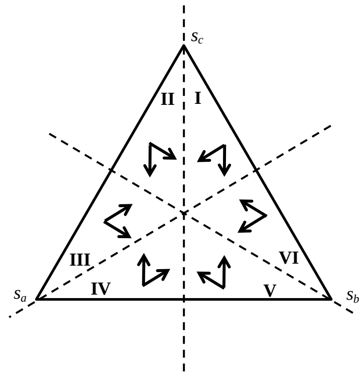


Figure 3: Direction of Dynamics

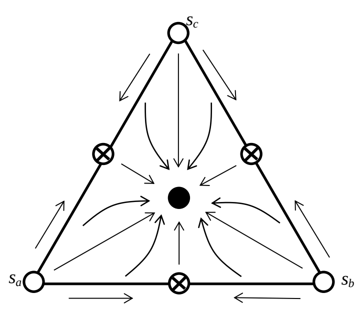


Figure 4: The Dynamics of Game

Consider the Buyer-Seller game featured in section 3.5 of your textbook. For that game,

a. find own-population effects (described by 2x2 matrices, as in section 3.7) so that the closed loop trajectories in the original game become inward spirals.
b. Find different own-population effects to that the trajectories become outward spirals.

Let the matrices for the game:

$u = \begin{pmatrix} 1 & -1 \\ 4 & 2 \end{pmatrix}$
$v = \begin{pmatrix} -8 & 2 \\ 0 & 1 \end{pmatrix}$

The generic formulation for own effects is decribed as:

$\Delta u = (1, -1)(u_a * p + u * q)$
$\Delta v = (1, -1)(v_a * q + u * p)$

Then the algebra would be:
$\Delta u = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} a & -a \\ 4a & 2a \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$

$\Delta u = -3a - 3$
and
$\Delta v = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -8a & 2a \\ 0a & 1a \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -8 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$

$\Delta v = -9aq_1 + a - 9p_1 + 1$
For $a > 1$, i.e., within the presence of strong own effects, the closed loop trajectories in the original game become inward spirals.

Note from Instructor: An easier way to answer this is to use very simple matrices, not those that look like the H-D example from the book. An answer from a previous class follows.

Recalling the payoff matrix of the buyers, $\mathbf{w} = \begin{pmatrix} 1 & -1 \\ -4 & 2 \end{pmatrix}$ and the one of the seller $\mathbf{u} = \begin{pmatrix} -8 & 2 \\ 0 & 1 \end{pmatrix}$. Moreover, let the matrix **s** and **r** describe respectively the share distribution of the buyer and the seller, and the parameters b and a denoting the potential presence of own-population effect respectively for the buyers and the sellers. Formally, $\mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$, $bs = \begin{pmatrix} bs_1 \\ bs_2 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ and $a\mathbf{r} = \begin{pmatrix} ar_1 \\ ar_2 \end{pmatrix}$.

From here, it is possible to derive the payoff of the buyers playing strategy 1 as

$$w_1 = \begin{pmatrix} 1 & -1 \end{pmatrix} \left(\begin{pmatrix} bs_1 \\ bs_2 \end{pmatrix} + \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \right) = bs_1 + r_1 - bs_2 - r_2$$

Similarly for the buyers playing strategy 2, the sellers playing strategy 1 and the sellers playing strategy 2,

$$w_2 = \begin{pmatrix} -4 & 2 \end{pmatrix} \left(\begin{pmatrix} bs_1 \\ bs_2 \end{pmatrix} + \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \right) = -4b_{s1} - 4r_1 + 2bs_2 + 2r_2$$

$$u_1 = \begin{pmatrix} -8 & 2 \end{pmatrix} \left(\begin{pmatrix} ar_1 \\ ar_2 \end{pmatrix} + \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \right) = -8ar_1 - 8s_1 + 2ar_2 + 2s_2$$

$$u_2 = \begin{pmatrix} 0 & 1 \end{pmatrix} \left(\begin{pmatrix} ar_1 \\ ar_2 \end{pmatrix} + \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \right) = ar_2 + s_2$$

Note that if $a = b = 0$, the expected payoffs are the same as the ones in the classical buyer-seller game.

Plotting those equations into R and using the differential equations solver one can see the different impact of a and b on the game equilibria. Here it is easy to get inward spirals for small $a > 0 = b$. It seems that $b > 0$ introduces new attractors at the corners.

Question 3

State spaces of dimension 4 include the tesseract (or hyper-cube) and the 5-simplex. There are three other possible state spaces. For each of the 2+3 = 5 possible state spaces of dimension 4, write down the number of distinct populations, the number of distinct strategies, and give a possible example.

The 5-simplex in dimension has its space state spanned by the vertices $e_1 = (1, 0, 0, 0, 0)$, $e_2 = (0, 1, 0, 0, 0)$, $e_3 = (0, 0, 1, 0, 0)$, $e_4 = (0, 0, 0, 1, 0)$ and $e_5 = (0, 0, 0, 0, 1)$
The state $s = (s_1, \ldots, s_5) = \sum_{i=1}^{5} s_i e_i$ is a mix of these five strategies.
5 cases for the 4-D state space:

- 5-simplex: 1 population, 5 strategies
- Cartesian product of a triangle and a square: 3 populations, 1 with 3 strategies, 1 with 2 strategies, and 1 with 2 strategies
- Cartesian product of a tetrahedron and a segment: 2 populations, 1 with 4 strategies and the other with 2 strategies
- Cartesian product of two triangles: 2 populations, each with 3 strategies
- Hypercube: 4 populations, each has 2 strategies

Figure 5: Cartesian Product Two Triangles

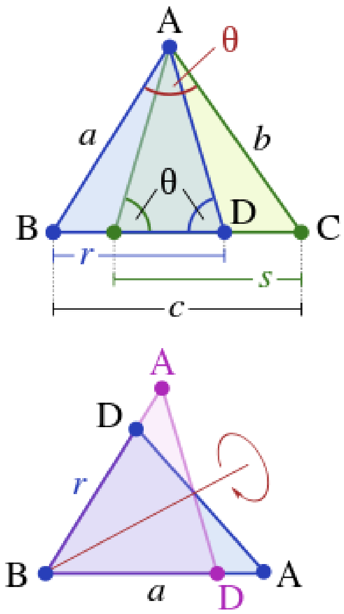


Figure 6: 5-simplex

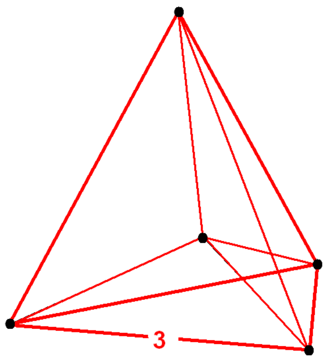


Figure 7: Hypercube