Equations for Competitive Markets

**Linear Demand:** \( q_d = a - bp \)  
**Linear Supply:** \( q_s = x + yp \)

**Log-linear demand:** \( \ln(q_d) = \ln(a) + \varepsilon_d \ln p \)  
**Log-Linear Supply:** \( \ln(q_s) = \ln(x) + \varepsilon_s \ln p \)

**Total Surplus=** Consumer Surplus + Producer Surplus;  
**Revenue=** Producer Surplus + Variable Cost

**Total Cost=** Fixed Cost + Variable Cost;  
**Profit=** Revenue - Total Cost = Producer Surplus - Fixed Cost

**Quantity Tax** (tax per unit): \( p_d = p_x + t \);  
**Value Tax** (tax on percentage spent): \( p_d = (1 + t)p_x \)

**Price Elasticity of Demand:** \( \varepsilon_d = \frac{\partial \ln(D)}{\partial \ln(p)} = \frac{\partial D}{\partial p} D \);  
If \( |\varepsilon| > 1 \) then curve is elastic.

**Tax Incidence Formula:** \( p_s(t) = p^* - \frac{\varepsilon D'}{S_D' + D'} \);  
\( p_d = p^* + \frac{\varepsilon S'}{S_D' + D'} \);  
If \( \varepsilon_d \) is constant: \( \frac{\partial p_d}{\partial t} = \frac{\varepsilon_s}{\varepsilon_s + \varepsilon_d} \)

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**Equations for Consumer Choice and Demand**

**Marginal Utility:** \( MU_i = \frac{\partial u}{\partial x_i} \);  
**Marginal Rate of Substitution:** \( MRS_{ij} = \frac{MU_i}{MU_j} \) and at interior optimum is \( \frac{p_i}{p_j} \)

**Perfect Substitutes:** \( u(x_1, x_2) = x_1 + cx_2 \);  
**Cobb-Douglas:** \( u(x_1, x_2) = \ln(x_1) + c \ln(x_2) \)

**CES Utility:** \( u(x_1, x_2) = \frac{1}{\rho} \ln(x_1^\rho + x_2^\rho); \rho \in (-\infty, 1] \);  
**Quasilinear:** \( u(x_0, x_1) = x_0 + g(x_1) \)

**Marshallian Demand:** \( x^* = (x^*_i(p, m), x^*_2(p, m), \ldots) \) is the solution to \( \max_{x \geq 0} u(x) \) s.t. \( m = p \cdot x \).  
The Lagrangian is \( L = u(x) + \lambda (m - p \cdot x) \).  
The FOCs can be written \( MU_i = \lambda p_i \) or \( MRS_{ij} = \frac{p_i}{p_j} \).

The solutions \( x^*_i(p, m) \) are homogeneous degree 0.

**Hicksian Demand:** \( h^*_i(p, u_0) : \min p \cdot x \) s.t. \( u(x) \geq u_0 \)

**Roy’s Identity:** \( x^*_i(p, m) = \frac{\varepsilon_m \partial u}{\partial p_i} \).

**Slutsky Equation:**  
\[ \frac{\partial x_i(p, m)}{\partial p_i} = \frac{\partial h_i(p, m)}{\partial p_i} - \frac{\partial x^*_i(p, m)}{\partial p_i} \]  
(Elasticity Form) \( \varepsilon_i = \varepsilon^H_i - \varepsilon_m \) for \( s_i = \frac{p_i x_i}{m} \)

**Demand Elasticity identity** for product i: \( \varepsilon_{in} + \varepsilon_{ii} + \sum_{j \neq i} \varepsilon_{ij} = 0 \)

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**Equations for Cost and Technology**

**Technical Rate of Substitution:** \( TRS_{ij} = -\frac{\partial f(x)}{\partial x_j} \bigg|_{x = 0} = \frac{m_j}{m_i} < 0; \)  
**MC:** \( MC(y) = \frac{\partial c}{\partial y} = \frac{\partial c}{\partial y} \) and \( \int MC = VC \).

**Factor Prices:** \( w = (w_1, w_2, \ldots, w_n) \);  
**Production Function:** \( y = f(x_1, x_2) \)

**Cost Function** with two factors: \( c(w, y) = w_1 x_1^1(w_1, w_2, y) + w_2 x_2^2(w_1, w_2, y) = \min_{x_1, x_2 \geq 0} w_1 x_1 + w_2 x_2 \) s.t. \( y = f(x_1, x_2) \)

**Shepard’s Lemma** conditional factor demand: \( x^*_i(w, y) = \frac{\partial c(w, y)}{\partial w_i} \)

**Learning Curve:** The typical specification is for \( Y_t = \sum_{t \leq T} \alpha_t \), AC falls proportionally, \( \ln AC_t = AC_0 - b \ln Y_t \)

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**Equations for Competitive Firms**

**SR Profit Maximization:** \( \max_{y, x \geq 0} \pi = \max_{y \geq 0} \max_{x \geq 0} R(y) - w_i x_i - w_f x_f \) s.t. \( y = f(x, \bar{x}) \) \[ \max[\pi] \] \( R(y) = \pi - c(y) \)

**Revenue if firm is competitive:** \( R(y) = py = pf(x, \bar{x}) \)  
FOC of unconditional factor demand: \( p \frac{\partial f(x, \bar{x})}{\partial p} = w_v \)

**Hotelling’s Lemma, Supply:** \( y^*(p, w) = \frac{\partial c(p, w)}{\partial p} \);  
unconditional factor demands: \( x_i(p, w) = -\frac{\partial c(p, w)}{\partial w_i} \)

**Shutdown Condition (Competitive Firms):** \( -F > py - c_v(y) - F \Rightarrow AVC = c_v(y) > p \)

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**Equations for Risky Choice**

Given a lottery with monetary outcomes \( m_1, \ldots, m_n \) and corresponding probabilities \( p_1, \ldots, p_n \), its **expected value** is \( Em = \sum p_i m_i \) and its **variance** is \( Var m = \sigma^2 m = E(m - Em)^2 = \sum p_i (m_i - Em)^2 \).

Given Bernoulli function \( u(m) \) — so \( u' > 0 \) and, if the person is risk-averse, \( u'' < 0 \) —  
the **certainty equivalent** \( m_{CE} \) of the lottery solves \( u(m_{CE}) = Eu(m) = \sum p_i u(m_i) \).

The **coefficient of absolute risk aversion** is \( a(m) = -u''(m)/u'(m) \) and the **coefficient of relative risk aversion** is \( v(m) = ma(m) \).

The **risk premium** is \( RP = Em - m_{CE} \). It is also given by the second term of the Taylor expansion of \( u \) around \( Em \).
Bayes Theorem: \( p(s|m) = \frac{p(m)s}{\sum_{t \in S} p(m|t)p(t)} \) or \( \frac{p(m)s}{p(t|m)} = \left[ \frac{p(m)s}{p(m)} \right] \left[ \frac{p(t)}{p(t|m)} \right] \) or \( \ln \) \text{prior odds} = \( \ln \) \text{likelihood ratio} + \( \ln \) \text{prior odds}. Note: \( s = \text{state}, m = \text{message} \).

Variance of Portfolio: Two assets: \( Var(p) = x_1^2 \text{var}(k) + 2x_1x_2 \text{Cov}(k, h) + x_2^2 \text{var}(h) \). Multiple assets: \( Var(p) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_ix_j\sigma_i\sigma_j \).

Equations for Monopolies

FOC for a monopolist: \( p(y) + p'(y)y = c'(y) \) which can be rewritten as \( p = \frac{1}{1+\frac{e}{x}} \text{MC} \); valid if \( \varepsilon < -1 \).

Passing Along Costs: \( \frac{\partial p}{\partial e} = \frac{1}{x+yp'(y)p'(y)} \).

Price Discrimination. Third Degree: \( \max p_1(x_1)x_1 - cx_1 + p_2(x_2)x_2 - cx_2 \). FOC gives: \( p_i(x_i) = \frac{1}{|x_i|} = c \) where \( \epsilon_i \) is the elasticity of demand in market \( i = 1, 2 \). The price-cost ratio gives the Markup factor: \( M_i = \frac{1}{1-\frac{\epsilon_i}{|x_i|}} \).

Cournot. Given inverse demand \( p(Y) = a - bY \), where \( Y = y_i + Y_- = \sum_{j=1}^{n} y_j \).

\( BR_i(Y_-) = \text{argmax } \pi_i = p(Y)y_i - c(y_i) \implies p(Y) + p'(Y)y_i - MC_i(y_i) = 0 \).

Nash equilibrium is where BR functions intersect: \( \Rightarrow \text{NE}_{\text{Cournot}} : y^* = \frac{N}{(N+1)b} (a - c) \implies y_i^* = \frac{a - c}{(N+1)b} \).

Bertrand. For firms with homogeneous goods, \( p = MC \) if equal MC, and \( p = \text{second lowest MC} \) if firms differ.

Stackelberg. Leader solves \( \max \pi_L(y_L, BR_F(y_L)) = p(y_L + BR_F(y_L))y_L - c(y_L) \).

Kinked Curves. Market demand: \( \text{MR} = \frac{dy}{dx} \) where \( \bar{p} \) is the established price. Rivals will match \( p < \bar{p} \) and will not match \( p > \bar{p} \). Profit maximization \( \Rightarrow \) not changing quantity (or price) as long as \( MC_{low} < MC < MC_{high} \) where \( MC_{low} = \text{MR}_{\text{match}}(\bar{p}) \), and \( MC_{high} = \text{MR}_{\text{no-match}}(\bar{p}) \).

Conjectural Variations. If \( p(Y) = p(y_1 + y_{-1}) \), then firm 1’s FOC is \( c'(y_1) = p(Y) + y_1p'(Y)[1 + \nu] \). The conjectural variation \( \nu = \frac{dy_1}{dy_{-1}} \) is 0 for Cournot, -.5 for Stackelberg leader (in linear duopoly), -1.0 in Bertrand, and \( \nu = \frac{y_{-1}}{y_1} \) in collusion/Cartel.

Hotelling location models. Duopoly case on \( [0, 1] \): Firm \( i \)’s BR to location choice \( z_j < .5 \) is \( z_i = z_j + \epsilon \), and to \( z_j > .5 \) is \( z_i = z_j - \epsilon \). Unique NE will be back to back at \( z = .5 \). Delivered Price for firm \( j \) at location \( z \) is \( p_j(z) = p + t|z - z_j| \).

Monopolistic Competition. Solve standard monopoly problem \( MR = MC \) and \( p = D^{-1}(q^*) \). Determine whether economic profits are \( > 0 \) or \( < 0 \). In LR equilibrium \( \pi = 0 \) since \( LRAC = LRAR \).

Equations for Intertemporal Equilibrium Theory

Slope of indifference curve \( \frac{\partial U}{\partial t} = MRS_{01} = 1 + MRTP \), where \( \partial U = \frac{\partial U}{\partial t} \) is the marginal utilities from the current and future consumptions, \( t = 0 \). And \( MRTP \), or the marginal rate of time preference, is defined as \( MRTP = \frac{\partial U}{\partial t} - 1 \).

Return on investment \( ROI = f(x) - x \). \( AROI = \frac{f(x)}{x} - 1 \). \( MROI = f'(x) - 1 \), or, in geometrical terms, slope of PPF= \( MR = 1 + MROI = f'(x) \).

Slope of budget line \( -\text{slope} = \frac{\partial P}{\partial x} = 1 + r \) where \( r \) is the real interest rate.

Wealth and present value. Given \( r > 0 \) and consumption stream \( C = (c_0, c_1) \), the present value of \( C \) is the horizontal axis intercept \( w \) of the budget line thru \( C \): \( w = PV_r(C) = c_0 + \frac{c_1}{1+rt} \), where \( w \) is called wealth in the intertemporal context.

Optimum investment. Given production function \( f \), endowment \( E = (e_0, e_1) \) and \( r \), the choice problem is \( \max_w w = PV_r(q) = q_0 + \frac{q_1}{1+rt} = e_0 - x + \frac{c_1 + f(x)}{1+rt} \). FOC gives \( r = MROI \).

Optimal individual borrowing, lending, and consumption. Given production stream \( Q \), utility \( U(c_0, c_1) \) and \( r \), max\( b(U(q_0 + b, q_1) - (1 + r)b) \) where \( b = c_0 - q_0 \) is the amount borrow. FOC gives \( r = MRTP \).

Fisher's equation \( k = r + \pi \).