ECON 204B Final Cheat Sheet

DECISION THEORY

Expected Utility Hypothesis
$\max_{a \in A} E_\pi u = \max_{a \in A} \sum_{s \in S} \pi(s) u[x(a,s)]$
CRRA: $u(x|r) = \frac{x^{1-r}}{1-r}$
CARA: $u(x|a) = 1 - e^{-ax}$

Coefficient of Absolute Risk Aversion
$A(x) = -\frac{u\prime(x)}{u''(x)}$, $x \in (-\infty, \infty)$

Coefficient of Relative Risk Aversion
$R(x) = \frac{-u''(x)}{u'(x)} x = xA(x)$
Risk Measurement
$Var_L[x] = E_L(x - E_L x)^2 = \sum_{i=1}^N p_i (x_i - \bar{x})^2 = E_L(x^2) - E_L(x)^2$
$\bar{x} = E_L x = \sum_{i=1}^N p_i x_i$
$\sigma_L = \sqrt{\text{Var}_L}$

Stochastic Dominance $F(x) \geq G(x) \quad \forall x \Rightarrow$ G FOSDs F
$\mu_F = \mu_G, \int_{-\infty}^x F(t)dt \geq \int_{-\infty}^x G(t)dt \forall x \Rightarrow$ G SOSDs F

Definitions and Conditional Probabilities
Prior prob. of s: $p(s) = \sum_{z \in Z} p(s,z)$
Prob. of message z: $p(z) = \sum_{s \in S} p(s,z)$
Likelihood: $p(z|s) = \frac{p(s,z)}{p(s)}$
Posterior prob. of s: $p(s|z) = \frac{p(s,z)}{p(z)}$
Bayes Theorem
$p(s|z) = \frac{p(z|s)p(s)}{p(z)}$
$p(s|z) = \frac{p(z|s)p(s)}{\sum_{t \in S} p(z|t)p(t))}$
$\frac{p(s|z)}{p(t|z)} = \frac{p(z|s)p(s)}{p(z|t)p(t)}$
$\ln(\frac{p(s|z)}{p(t|z)}) = \ln(\frac{p(z|s)}{p(z|t)}) + \ln \frac{p(s)}{p(t)}$

Value of Information
$VI = \sum_{z \in Z} p(z) \sum_{s \in S} p(s|z)[u_z^*(s) - u_0^*(s)]$ (Risk Neutral)
solves $0 = \sum_{z \in Z} \sum_{s \in S} p(z)p(s|z)u(a_z^* - x, s)$ (General Case)

Dynamic Programming $V(y_0, 0) = \underset{a_t \in A_t}{\text{maximize}} \quad \sum_{t=0}^T F(a_t, y_t, t)$
s.t $\quad y_{t+1} = y_t + Q(a_t, y_t, t) \qquad G(a_t, y_t, t) \geq 0$

GAME THEORY BASICS

Strategy s_i is *weakly dominant* if:
$f_i(s_i, s_{-i}) \geq f_i(s_i', s_{-i}) \quad \forall s_i' \in S_i, \quad s_{-i} \in S_{-i}$.
A pure strategy $s_i \in B_i(s_i)$ is a *best response* to profile s_{-i} if:
$f_i(s_i, s_{-i}) \geq f_i(s_i', s_{-i}) \quad \forall s_i' \in S_i$.
Nash Equilibrium
A profile $s_i^* = (s_1^*, ..., s_n^*)$ is a Nash Equilibrium if:
$s_i^* \in B(s_{-i}^*) \quad i = 1, ..., n)$.
Payoff Function for Mixed NE
Player i's expected payoff when other players' pure strategy is s_{-i}:

$$f_i(\sigma_i, s_{-i}) = \sum_{k=1}^N p_k f_i(t_k, s_{-i})$$

Player i's expected payoff when playing own pure strategy t_i:

$$f_i(t_k, \sigma_{-i}) = \sum_{j=1}^m q_j f_i(t_k, s_{-i}^j)$$

This is used to find mixed NE. For example, in the 2×2 case:

$$f_1(s_1, \sigma_{-1}) = f_1(s_2, \sigma_{-1})$$

$$f_2(s_1, \sigma_{-2}) = f_2(s_2, \sigma_{-2})$$

EXTENSIVE FORM GAMES

Procedure for finding solution of EFGs with perfect information

1. Convert penultimate nodes ν into terminal nodes
   (a) If ν is owned by player i, then he chooses the maximum payoff
   (b) If ν is owned by Nature, take expectation over the payoff vector
2. Iterate over step 1 until reaching the initial node
3. Reconstruct each player's strategy for their choices in steps 1-2
4. The resulting profile is subgame perfect Nash equilibrium (SPNE)
5. In the case if *imperfect information*, find the smallest subgame that contains terminal nodes. Find all NE of that subgame. Replace the initial node of that subgame by a NE payoff vector. Iterate to a solution and get one SPNE. Then loop on step 1 until all N equilibria in the minimal game have been used.

BNE, PBE, AND SE

1. Beliefs μ_i at each info set for player i are consistent with *common prior* and likelihood from s_{-i}^* and own realized type $\bar{\theta}_i$ via Bayes
2. At each info set, player i maximizes $E(u_i|\mu_i)$:
   $E_{\theta_i}[u_i(s_i^*(\bar{\theta}_i), s_{-i}^*(\theta_{-i}), \theta_i)|\bar{\theta}_i \geq E_{\theta_i}[u_i(s_i'(\bar{\theta}_i), s_{-i}^*(\theta_{-i}), \theta_i)|\bar{\theta}_i]$
3. Previous items hold in every subgame
4. Robust to sufficiently small trembles

- 1 and 2 constitute a *Bayesian Nash Equilibrium*
- 1 - 3 constitute a *Perfect Bayesian NE*
- 1 - 4 constitute a *Sequential Equilibrium*

REPEATED GAMES

For finite T, use IDDS. For infinite T, with interest rate r and continuation probability $q \Rightarrow$ discount factor is $d = \frac{q}{1+r}$
Grim trigger: Revert to NE strategy if opponent deviates.

EVOLUTIONARY GAMES

2×2 games, 1 population.

$\pi(r,s) = rAs^T = (x, 1-x) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix}$

Write $D(p) = \underbrace{(a_{11} - a_{21})}_{a_1} p + \underbrace{(a_{12} - a_{22})}_{a_2}(1-p)$

HD type: $a_1, a_2 < 0$. Mixed NE $p^* = \frac{a_2}{a_1+a_2} \in (0,1)$, stable because $\partial D(p)/\partial p < 0$ (downcrossing)

CO type: $a_1, a_2 > 0$. Mixed NE $p^* = \frac{a_2}{a_1+a_2} \in (0,1)$, unstable because $\partial D(p)/\partial p > 0$ (upcrossing)

DS type: a_1 and a_2 have opposite signs. No mixed NE because $p^* = \frac{a_2}{a_1+a_2} \notin (0,1)$.

2×2 games, 2 populations:
Phase state is unit square rather than line.

Replicator dynamics: $\dot{s}_i = s_i(w_i(s) - \overline{w}(s))$

COOPERATIVE GAMES

Alternating offers game: $\delta \in (0,1) \Rightarrow$ In equilibrium, Player 1 obtains $\frac{1}{1-\delta}$, Player 2 obtains $\frac{\delta}{1-\delta}$.

Nash Bargaining Solution: With threat points $\underline{u}_1$ and $\underline{u}_2$: $(u_1^*, u_2^*) = \arg\max_{(u_1,u_2)\in X}(u_1 - \underline{u}_1)(u_2 - \underline{u}_2)$.

A characteristic function is a mapping $v : 2n \to \mathbb{R}$ such that
(1) $v(\emptyset) = 0$, (2) $v(K) \le v(N) < \infty \; \forall K \in 2^n$,
(3) $v(K) + v(L) \le v(K \cup L) \; \forall$ disjoint $K, L \in 2^n$.

Feasibility with transferable utility implies
$x = (x_1, \ldots, x_n) \in X \Rightarrow \sum_{i=1}^N x_i \le v(N)$.

A payoff vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is individually rational if $x_i \ge v(\{i\}) \; \forall i \in N$, Pareto optimal if $\sum_{i \in N} x_i \ge v(N)$.

An imputation x is individually rational and Pareto optimal.

If $\sum_{i \in K} x_i < v(K)$ then x is blocked by K.

Core consists of all imputations not blocked by a coalition.

Shapley Value: $\phi_i(v) = \frac{1}{n!}\sum_\rho MC_i(\rho)$, ρ a coalition formation sequence, $MC_i(\rho)$ marginal contribution when joining.

A CFG is convex if $v(K \cup L) + v(K \cap L) \ge v(K) + v(L) \; \forall K, L \subseteq N$.

IMPERFECT COMPETITION

Monopolist's FOC is $q_m[p'(q_m)] + p(q_m) = c'(q_m)$

DWL is $\int_{q_m}^{q_0} [p(z) - c'(z)]\, dz$

Bertrand Competition: Firms simultaneously choose price to maximize profit $\pi_j(p_j, p_k) = x_j(p_j, p_k)[p_j - c]$. The unique NE is $p_j = p_k = c, \pi_j = \pi_k = 0$.

Cournot Competition: Firms simultaneously choose quantity to maximize profit $\pi_j(q_j, q_k) = P(q_j + q_k)q_j - cq_j$.
The FOCs are: $P'(q_j + q_k)q_j + P(q_j + q_k) = c$.

Hotelling model: In the duopoly where firms choose location but not price and $p_1 = p_2 = p > c = c_1 = c_2$, the unique NE is for both firms to locate at the middle point.

Conjectural Variations: $c'(q_j) = P(Q) + q_j P'(Q)\left[\frac{dq_j}{dq_j} + \frac{dq_k}{dq_j}\right] = $
$= P(Q) + q_j P'(Q)(1 + \nu)$

ASYMMETRIC INFORMATION

A C.E. with asymmetric information is (p^*, Θ^*) s.t.
(1) participants are $\Theta^* = \{\theta : r(\theta) \le p^*\}$,
(2) price clears the market: $p^* = \mathbb{E}(\theta | \theta \in \Theta^*) = E(\theta | p^*)$

A PBE in a signaling game is $\{m^*(\theta), a^*(m), \mu(\theta|m)\}$ such that
(1) $m^*(\theta) = \arg\max_{m \in M} \pi_s(m, a^*(m), \theta)$,
(2) $a^*(m) = \arg\max_{a \in A} \sum_{\theta \in \Theta} \mu(\theta|m)\pi_R(a, m, \theta)$,
(3) Beliefs $\mu(\theta|m)$ are consistent with Bayes' Rule.

Pooling equilibrium: All types send same message m. Posterior equals prior if message is sent, else posterior is free parameter. No profitable deviation for any type.

Separating equilibrium: Each type sends different message. Respondent infers type with certainty from message. No profitable deviation any type, given respondent's beliefs.

Screening: Uninformed party offers menu of choices $\{A, B, ...\}$. In separating PBE, type is revealed by self-selection, e.g., if the following hold:
1. I.C.: $\mathbb{E}(U_{\theta_H}(A)) \ge \mathbb{E}(U_{\theta_H}(B))$, $\mathbb{E}(U_{\theta_L}(B)) \ge \mathbb{E}(U_{\theta_L}(A))$
2. P.C.: $\mathbb{E}(U_{\theta_H}(A)) \ge r(\theta_H)$, $\mathbb{E}(U_{\theta_L}(B)) \ge r(\theta_L)$

PRINCIPAL/AGENT PROBLEM

In 2-level case, effort $e \in E = \{e_L, e_H\}$, profit $\pi \in \Pi = [\underline{\pi}, \overline{\pi}]$, pdf $f(\pi|e_H)$ FOSDs $f(\pi|e_L)$.
Agent's payoff is $U_A(w, e) = u(w) - g(e)$; with $g(e_H) > g(e_L)$; $u'(w) > 0$, $u''(w) \le 0$; reservation utility $\overline{u}$.

Principal chooses wage schedule to maximize profit $\int_{\underline{\pi}}^{\overline{\pi}} [\pi - w(\pi)] f(\pi|e) d\pi$ (min $E(w) = \int_{\underline{\pi}}^{\overline{\pi}} w(\pi) f(\pi|e) d\pi$) s.t.
(1) P.C. [multiplier γ]: $\int_{\underline{\pi}}^{\overline{\pi}} u(w(\pi)) f(\pi|e) d\pi - g(e) \ge \overline{u}$.
(2) I.C. [multiplier μ]:

$$\int_{\underline{\pi}}^{\overline{\pi}} u(w(\pi)) f(\pi|e) d\pi - g(e) \ge \int_{\underline{\pi}}^{\overline{\pi}} u(w(\pi)) f(\pi|\tilde{e}) d\pi - g(\tilde{e})$$

For risk-averse agent with effort unobservable, the wage schedule satisfies

$$\frac{1}{u'(w(\pi))} = \gamma + \mu \left[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}\right]$$