Answer Key for PS 1

thanks to Zhaoqi and Gregory

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Problem #1

Directly calculating the expected utility for an arbitrary $a \in \mathbb{R}$ yields a)

$$E[1-(a-s)^2] = \int_{-\infty}^{\infty}(1-(a-s)^2)dF(s) = 1 - \int_0^1 (a-s)^2 ds =$$

$$= 1 + \frac{(a-s)^3}{3}|_{s=0}^{s=1} = 1 + \frac{(a-1)^3}{3} - \frac{a^3}{3}$$

If there is an interior solution for the optimum, it must satisfy

$$\frac{\partial}{\partial a}E[1-(a-s)^2] = (a-1)^2 - a^2 = 0,$$

with solution

$$-2a + 1 = 0 \implies a^* = 1/2$$

Since a^* is an admissible value ($a^* \in [-10, 20]$, it is the optimal choice of a. Utility at $a = a^*$ is

$$u(a^*) = 1 + \frac{(-1/2)^3}{3} - \frac{(1/2)^3}{3} = \frac{11}{12}$$

b)

For an arbitrary distribution of s with differentiable pdf, the optimal value must satisfy

$$\frac{\partial}{\partial a}E[1-(a-s)^2] = E\left[\frac{\partial}{\partial a}(1-(a-s)^2)\right] = -2E[a-s] = -2a + 2E[s] = 0$$

$$\implies a^* = E[s],$$

provided $E[s] \in [-10, 20]$. (The Leibniz integral rule was used to move the differentiation inside the expectation.)

Expected utility at a^* is

$$u(a^*) = E[1-(s-E[s])^2] = 1 - Var[s]$$

Remark from Dan: the function $1-(a-s)^2$ is called the quadratic scoring rule, and it is sometimes used to elicit beliefs. If you want a forecast of, say, next year's GDP, you could tell the forecaster that her payment will be proportional to this function (where $a =$ her forecast and $s =$ the official GDP when it is reported), and it would be in her interest to think carefully and report her subjective expectation.

Problem #2

a)
Let's say you are only willing to pay \$850 for this lottery. To find a, solve $1 - e^{-850a} = \frac{1}{2}(1 - e^{-800a}) + \frac{1}{2}(1 - e^{-1200a})$. This must be solved numerically; the solution is $a \approx 0.0138$.

b) Calculate $\mu_L = \frac{1}{2}(800) + \frac{1}{2}(1200) = 1200$ and $\sigma_L^2 = \frac{1}{2}(800-1000)^2 + \frac{1}{2}(1200-1000)^2 = 40000$.
The mean-variance approximation to the utility that gives the same certainty equivalent for this lottery satisfies $\mu_L - c\sigma_L^2 = 850$, Solving for c yields $c = 150/40000 = 0.00375$

c) We have $\mu_M = 1$ and $\sigma_M^2 = (0.001)999^2 + (0.999)(-1)^2 \approx 999$

Given our CARA preferences, the expected utility from lottery M is $0.001(1 - e^{-0.0138 \cdot 1000}) + 0.999(1 - e^{-0.0138 \cdot 0}) = 0.001(1 - e^{-13.8}) > 0$

Using the mean-variance approximation to utility from part b, we have $\mu_M = 1 - 0.00375 \cdot 999 = -2.75 < 0$

d) Lottery M first order stochastic dominates the lottery [0 with certainty], since M guarantees at least as high a payoff in every possible state. Then, because the utility function is increasing, CARA preferences tell us that M is preferred. (So does common sense.) However, mean-variance utility for [0 with certainty] is $0 - c0 = 0$, which is larger than mean-variance for M. So according to mean-variance preferences, [0 with certainty] is preferred! The mean-variance approximation is not good here because the remainder term (proportional to $(x - \mu_M)^3$) in the Taylor expansion is large in this case, due to the outlying outcome with payoff \$1000.

Problem #3

a)
The optimization problem is to maximize $.5\frac{x^{1-\gamma}}{1-\gamma} + .5\frac{y^{1-\gamma}}{1-\gamma}$ subject to $3x + 7y = 35$. The FOCs for x and y are $0.5x^{-\gamma} - 3\lambda = 0$, and $0.5y^{-\gamma} - 7\lambda = 0$, where λ is a Lagrange multiplier. Dividing these FOCs yields $3/7 = (x^*/y^*)^{-\gamma}$. We are given $x^* = 7$ and $y^* = 2$, so we have $\gamma = -\ln(3/7)/\ln(7/2) \approx 0.676$.

b)
In general, the optimization problem is to maximize

$$u(x, y) = \pi_X \left(\frac{x^{1-\gamma}}{1-\gamma} \right) + (1 - \pi_X) \left(\frac{y^{1-\gamma}}{1-\gamma} \right)$$

subject to $p_x x + p_y y = M$. Using a Lagrange multiplier λ, the FOCs are

$$\pi_X x^{-\gamma} = \lambda p_x$$

$$(1 - \pi_X) y^{-\gamma} = \lambda p_y$$

$$p_x x + p_y y = M$$

Dividing the first two FOCs yields

$$\frac{1-\pi_X}{\pi_X}\frac{p_x}{p_y} = \left(\frac{y}{x}\right)^\gamma$$

Solving for γ yields

$$\gamma^* = \frac{\ln\left(\frac{1-\pi_X}{\pi_X}\frac{p_x}{p_y}\right)}{\ln\frac{y}{x}}$$

Remark from Dan: Some recent literature uses exactly this approach to estimate a person's coefficient of absolute risk aversion.

Problem #4

a)Assuming that the tests are conditionally independent,
$P(A, t_1^{\pm}, t_2^{\pm}) = P(t_1^{\pm}|\ A)P(t_2^{\pm}|\ A)P_A$
$\Rightarrow P(t_1^{+}|\ A)P(t_2^{+}|\ A)P_A = 0.084$
$\Rightarrow P(t_1^{+}|\ A)P(t_2^{-}|\ A)P_A = 0.336$
$\Rightarrow P(t_1^{-}|\ A)P(t_2^{+}|\ A)P_A = 0.036$
$\Rightarrow P(t_1^{-}|\ A)P(t_2^{-}|\ A)P_A = 0.144$
Likewise:
$P(B, t_1^{\pm}, t_2^{\pm}) = P(t_1^{\pm}|\ B)P(t_2^{\pm}|\ B)P_B$
$\Rightarrow P(t_1^{+}|\ B)P(t_2^{+}|\ B)P_B = 0.08$
$\Rightarrow P(t_1^{+}|\ B)P(t_2^{-}|\ B)P_B = 0.08$
$\Rightarrow P(t_1^{-}|\ B)P(t_2^{+}|\ B)P_B = 0.12$
$\Rightarrow P(t_1^{-}|\ B)P(t_2^{-}|\ B)P_B = 0.12$
b) The prior probabilities of the diseases are their relative incidences in the population of patients. The message probabilities are

$$P[t1 = +] = P[t1 = +|A]P[A] + P[t1 = +|B]P[B] = (0.7)(0.6) + (0.4)(0.4) = 0.58$$

$$P[t2 = +] = P[t2 = +|A]P[A] + P[t2 = +|B]P[B] = (0.2)(0.6) + (0.5)(0.4) = 0.32$$

c) Applying Bayes Theorem, version 1, we get
$P[A|t_1 = +] = \frac{P[t_1=+|A]P[A]}{P[t_1=+]} = \frac{(0.7)(0.6)}{0.58} = 0.724$,
$P[A|t_1 = -] = \frac{P[t_1=-|A]P[A]}{P[t_1=-]} = \frac{(0.3)(0.6)}{0.42} = 0.429$,
$P[A|t_2 = +] = \frac{P[t_2=+|A]P[A]}{P[t_2=+]} = \frac{(0.2)(0.6)}{0.32} = 0.375$, and
$P[A|t_2 = -] = \frac{P[t_2=-|A]P[A]}{P[t_2=-]} = \frac{(0.8)(0.6)}{0.68} = 0.709$

d) Again apply Bayes Theorem. There are many ways to do it; here is one way:
$P(A|\ B, C) = \frac{P(A,B,C)P(C)}{P(B,C)P(C)} = \frac{P(A,B|\ C)}{P(B|\ C)}$
$P(A|\ t_1^{\pm}, t_2^{\pm}) = \frac{P(A,t_1^{\pm},t_2^{\pm})}{P(t_1^{\pm},t_2^{\pm})}$
$P(t_1^{\pm}, t_2^{\pm}) = P(t_1^{\pm}, t_2^{\pm}|\ A)P(A) + P(t_1^{\pm}, t_2^{\pm}|\ B)P(B) = 0.084 + 0.08 = 0.164$
$\Rightarrow P(t_1^{+}, t_2^{+}) = 0.164$
$\Rightarrow P(t_1^{+}, t_2^{-}) = 0.416$
$\Rightarrow P(t_1^{-}, t_2^{+}) = 0.156$

$\Rightarrow P(t_1^-, t_2^-) = 0.264$

$$P(A|\ t_1^+, t_2^+) = \frac{P(A, t_1^+, t_2^+)}{P(t_1^+, t_2^+)} = 0.512$$
$$P(A|\ t_1^+, t_2^-) = 0.808$$
$$P(A|\ t_1^-, t_2^+) = 0.231$$
$$P(A|\ t_1^-, t_2^-) = 0.545$$

Problem #5

a)
Assume that the payoff if the salesman is not hired is zero. Given the prior probabilities, the expected payoff from hiring the salesman is $(0.1)(20) + (0.5)(5) - (0.4)(10) = 0.5 > 0$, so the salesman is hired. Given perfect information, a good or great salesman would be hired, and a poor salesman would not be hired. The dealership's decision is unchanged from having perfect information if the salesman is good or great, so the value of perfect information in those cases is zero. If the salesman is poor, the value of perfect information is 10, since having the information saves the dealership from a loss of 10. Hence, the expected value of perfect information is $(0.4)(10) = 4$.

b) The table below gives the expected payoff from hiring the salesman given the posterior probabilities associated with each possible observation of n for the first several values of n. The only negative payoff (implying that the salesman is not hired) is when no cars are sold. This occurs with probability

$$P(n = 0|poor)P(poor) + P(n = 0|good)P(good) + P(n = 0|great)P(great) =$$

$$= (0.4)(0.551) + (0.5)(0.418) + (0.1)(0.031) = 0.432$$

From the table, the expected payoff if no cars are sold is -2.808, so the gross value of this information is (0.432)(2.808)=1.213. Since it cost 0.04 to hire for a week, the net value of the information is 1.173.

=E$3*EXP(-E$2)*E$2^$C4/($E$3*EXP(-$E$2)*$E$2^$C4+$F$3*EXP(-$F$2)*$F$2^$C4+$G$3*EXP(-$G$2)*$G$2^$C4)

C	D	E	F	G	H	I	J	K	L	M	N
		poor	good	great							
	lambda*t	0.5	1	2							
n	prior	0.4	0.5	0.1	payoff						
0	posterior	0.551284	0.417964	0.030752	-2.80798						
1		0.365036	0.553514	0.081451	0.74622						
2		0.203039	0.615745	0.181216	4.672663						
3		0.094026	0.570295	0.33568	8.624808						
4		0.036482	0.442546	0.520972	12.26735						
5		0.012138	0.294495	0.693367	15.21842						
6		0.003597	0.174536	0.821867	17.27404						
7		0.000988	0.095895	0.903116	18.53192						
8		0.00026	0.050402	0.949339	19.23618						
9		6.66E-05	0.025858	0.974076	19.61014						
10		1.69E-05	0.013099	0.986884	19.80301						

c) The salesman is not fired since he or she has a positive expected payoff given that two cars are sold the first week. The dealership updates its prior probabilities, which are now the posterior probabilities from observing two cars sold. Now, the spreadsheet shows that one week is not sufficient to obtain any information that will change the dealership's decision; even if no cars are sold the second week, the expected payoff of the salesman is positive. Since it costs to hire on a temporary basis, the salesman is hired permanently.

Problem #6

Productivity old tech is $\sim N(10, 1)$, and Productivity new tech is $\sim N(12, 1)$.
Define state A: Rival has new tech. $P(A) = 0.1, P(\bar{A}) = 0.9$ are priors.
Rival reports productivity x.
Want to solve: $P(A|x) = 0.5$, i.e., find x such that posterior odds = .5/.5=1.0.
Likelihood ratio is the ratio of Normal densities $\phi(x|\mu, \sigma)$ evaluated at x.
Using Version 3 of Bayes Theorem, the posterior odds at the critical value of x are

$$\begin{aligned} 1 = \frac{P(A|x)}{P(\bar{A}|x)} &= \frac{\phi(x|A)}{\phi(x|\bar{A})}\frac{P(A)}{P(\bar{A})} \\ &= \frac{\frac{1}{\sqrt{2\pi}} exp[-\frac{(x-12)^2}{2}] \times 0.1}{\frac{1}{\sqrt{2\pi}} exp[-\frac{(x-10)^2}{2}] \times 0.9} \end{aligned}$$

$\implies exp[-\frac{(x-12)^2}{2}] = exp[-\frac{(x-10)^2}{2}] \times 9$
$\implies -\frac{(x-12)^2}{2} = -\frac{(x-10)^2}{2} + \ln 9 \implies x = (144-100)/4 + (2/4)\ln 9 = 11 + \ln 3 \approx 12.1.$
Therefore any observed productivity ≥ 12.1 implies that the rival has new technology with probability at least 0.5.

Problem #7

(a)There are two states:

$$ADEQUATE(A) \quad \text{and} \quad SUBSTANDARD(S)$$

and two actions:

$$adequate(a) \quad \text{and} \quad substandard(s).$$

The mean test outcome $\bar{X}$ of the test outcomes $\{X_1, X_2, ..., X_n\}$ is distributed as

$$\bar{X} \sim N(1, 9/n) \text{ if adequate (A), } \sim N(-1, 9/n) \text{ if substandard (S).}$$

Using backward induction (BI), we first consider the final choice, between adequate (a) and substandard (s) given the observed mean test outcome $\bar{X}$ for test size n. The expected payoff of a/s choice is

$$\begin{aligned} E[u(a|X,n)] &= -nP(A|\bar{X}) - (n+1000)P(S|\bar{X}) \\ E[u(s|X,n)] &= -nP(S|\bar{X}) - (n+1000)P(A|\bar{X}) \end{aligned}$$

Choosing adequate (a) is optimal, if

$$E[u(a|X,n)] = -nP(A|\bar{X}) - (n+1000)P(S|\bar{X}) \geq -nP(S|\bar{X}) - (n+1000)P(A|\bar{X}) = E[u(s|X,n)]$$

$$-n(P(A|\bar{X}) + P(S|\bar{X})) - 1000P(S|\bar{X}) \geq -n(P(S|\bar{X}) + P(A|\bar{X})) - 1000P(A|\bar{X})$$

$$P(S|\bar{X}) \geq P(A|\bar{X}),$$

which is, by Bayes' theorem, equivalent to

$$\frac{P(\bar{X}|S)P(S)}{P(\bar{X}|A)P(A) + P(\bar{X}|S)P(S)} \geq \frac{P(\bar{X}|A)P(A)}{P(\bar{X}|A)P(A) + P(\bar{X}|S)P(S)}$$

$$\Leftrightarrow P(\bar{X}|S) \geq P(\bar{X}|A) \quad \text{(given the prior probabilities).}$$

By using the pdf of Normal distribution, we get

$$exp\Big(-\frac{1}{2}\Big(\frac{\bar{X}-1}{3/\sqrt{n}}\Big)^2\Big) \geq exp\Big(-\frac{1}{2}\Big(\frac{\bar{X}+1}{3/\sqrt{n}}\Big)^2\Big)$$

$$(\bar{X}-1)^2 \leq (\bar{X}+1)^2$$

$$\bar{X} \geq 0.$$

Therefore, the decision rule is "choose a if $\bar{X} \geq 0$ and choose s otherwise."
(You might use symmetry to obtain this natural decision rule more quickly, but the long method used above shows how to generalize to non-symmetric priors and/or non-symmetric loss (or utility)

functions.)

Continuing BI, we now consider the strategy of the choice of how many units to test. The expected payoff is

$$\begin{aligned} E[u|n] &= P(\bar{X} \geq 0)\Big(-nP(A|\bar{X} \geq 0) - (n+1000)P(S|\bar{X} \geq 0)\Big) \\ &\quad +P(\bar{X} < 0)\Big(-nP(S|\bar{X} < 0) - (n+1000)P(A|\bar{X} < 0)\Big) \end{aligned}$$

By Bayes' theorem,

$$P(A|\bar{X} \geq 0) = \frac{P(\bar{X} \geq 0|A)P(A)}{P(\bar{X} \geq 0|A)P(A) + P(\bar{X} \geq 0|S)P(S)}$$
$$P(A|\bar{X} < 0) = \frac{P(\bar{X} < 0|A)P(A)}{P(\bar{X} < 0|A)P(A) + P(\bar{X} < 0|S)P(S)}$$

For the ease of derivation, express the probability using cdf of standardized normal distribution.

$$P(\bar{X} \geq 0|A) = P\big(z \equiv \frac{\bar{X}-1}{3/\sqrt{n}} \geq \frac{-1}{3/\sqrt{n}}\big) = 1 - \Phi\big(\frac{-1}{3/\sqrt{n}}\big)$$

Similarly,

$$\begin{aligned} P(\bar{X} < 0|A) &= \Phi\big(\frac{-1}{3/\sqrt{n}}\big) \\ P(\bar{X} \geq 0|S) &= 1 - \Phi\big(\frac{1}{3/\sqrt{n}}\big) \\ P(\bar{X} < 0|S) &= \Phi\big(\frac{1}{3/\sqrt{n}}\big) \end{aligned}$$

Since standard normal distribution is symmetric around 0,

$$\begin{aligned} P(\bar{X} \geq 0|A) &= P(\bar{X} < 0|S) \\ P(\bar{X} < 0|A) &= P(\bar{X} \geq 0|S) \end{aligned}$$

Therefore, the maximal expected utility given N simplifies to

$$\begin{aligned} E[u|n] &= -n - 1000\Phi\big(\frac{-1}{3/\sqrt{n}}\big) \\ &= -n - 1000\int_{\infty}^{\frac{-1}{3/\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \end{aligned}$$

The agent choose n to maximize the expected utility. Using the fundamental theorem of calculus and the chain rule, the F.O.C is

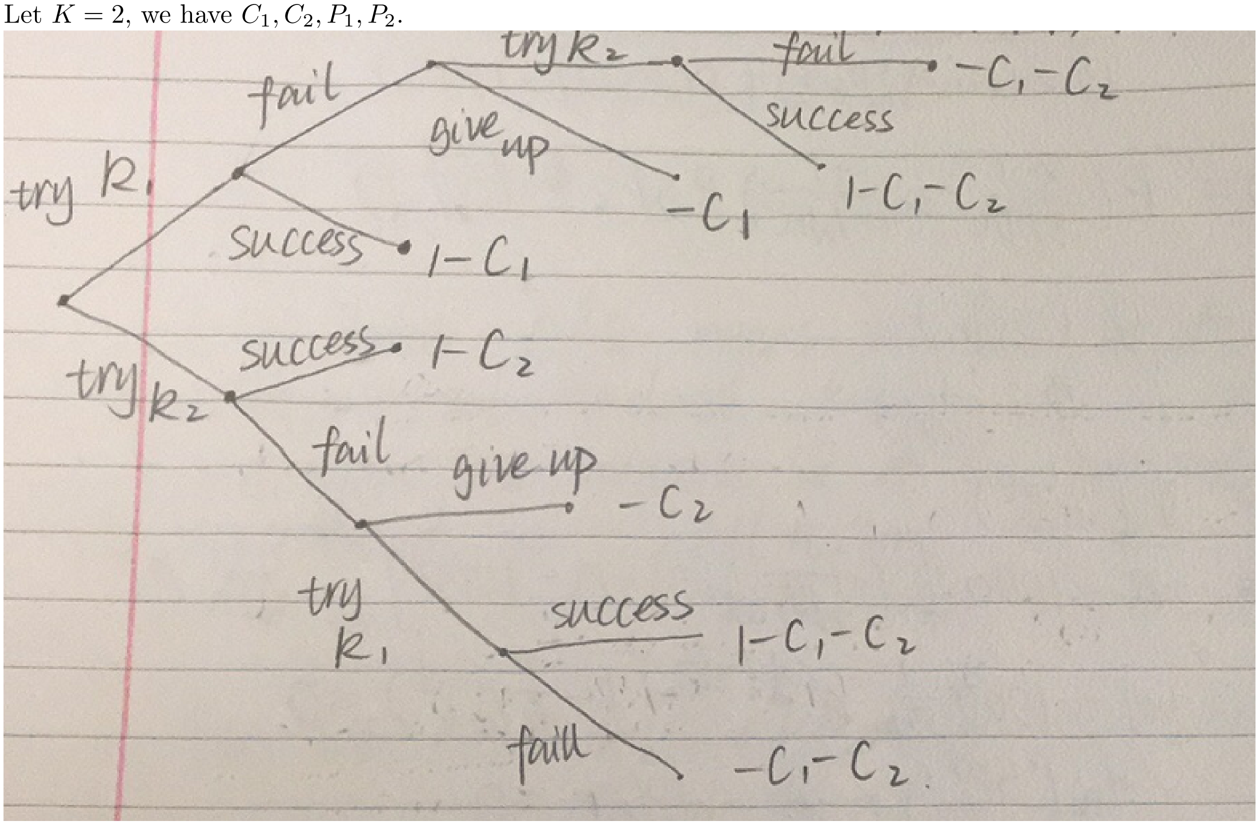
$$\begin{aligned}
\frac{dE[u|n]}{dn} = 0 \quad &\iff \quad -1 - 1000\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{-1}{3/\sqrt{n}}\right)^2}\left(-\frac{1}{6}n^{-\frac{1}{2}}\right) = 0 \\
&\iff \quad \frac{500}{3\sqrt{2\pi}} \cdot e^{-\frac{n}{18}} \cdot n^{-\frac{1}{2}} = 1 \\
&\iff \quad \ln\left(\frac{500}{3\sqrt{2\pi}}\right) - \frac{n}{18} - \frac{1}{2}\ln n = 0 \\
&\iff \quad \frac{n}{9} + \ln n = 2\ln\left(\frac{500}{3\sqrt{2\pi}}\right).
\end{aligned}$$

Solving numerically through a spreadsheet, we get the nearest integer to the solution as

$$n^* = 42.$$

Problem #8

Let $K = 2$, we have C_1, C_2, P_1, P_2.



If we try k_1 first.

$E(u_1) = P_1(1-C_1)+(1-P_1)P_2(1-C_1-C_2)+(1-P_1)(1-P_2)(-C_1-C_2) = P_1+P_2-C_1-C_2+P_1C_2$

If we try k_2 first.

$$E(u_2) = P_1 + P_2 - C_1 - C_2 + P_2C_1$$

So, if $E(u_1) > E(u_2), P_1C_2 > P_2C_1$, which means $\frac{P_1}{C_1} > \frac{P_2}{C_2}$, then we should try k_1 first.
If $-C_1 > P_2(1 - C_1 - C_2) + (1 - P_2)(-C_1 - C_2) \Rightarrow P_2 < C_2$
Then we should give up.
Thus, we should give up all methods with $\frac{P_i}{c_i} < 1$, then try each methods with descending order of $\frac{P_i}{C_i}$.