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1. Prove that every payoff matrix for a single population game has the same continuous dynamics as a "normalized" payoff matrix with zeros along the main diagonal.

It preserves both relative and absolute dynamics.

Proof. Replication dynamics for a system described by fitness matrix M are governed by the equation of motion (using the Einstein summation convention)

$$\dot{s}_i = s_i(e_k - s_k)M_{kj}s_j$$

or in relative form, as

$$\frac{d}{dt}\left(\ln\frac{s_i}{s_j}\right) = \frac{s_j}{s_i}\frac{d}{dt}\left(\frac{s_i}{s_j}\right) = M_{ik}s_k - M_{jk}s_k$$

Let us consider the alternate fitness matrix $A_{ij} = M_{ij} - M_{jj}\mathbf{j}_j\mathbf{j}_i$, where $\mathbf{j}$ is a vector of ones. Substituting A_{ij} into the equations above, the relative dynamics are given by

$$\begin{aligned}\frac{d}{dt}\left(\ln\frac{s_i}{s_j}\right) &= A_{ik}s_k - A_{jk}s_k \\ &= (M_{ik} - M_{kk}\mathbf{j}_k\mathbf{j}_i)s_k - (M_{jk} - M_{kk}\mathbf{j}_k\mathbf{j}_j)s_k \\ &= (M_{ik} - M_{jk} - ((\mathbf{j}_i - \mathbf{j}_j)M_{kk}\mathbf{j}_k)s_k) \\ &= M_{ik}s_k - M_{jk}s_k\end{aligned}$$

which is unchanged. The absolute dynamics are also preserved

$$\begin{aligned}\dot{s}_i &= s_i(e_k - s_k)A_{kj} \\ &= s_i(e_k - s_k)(M_{kj} - M_{jj}\mathbf{j}_j\mathbf{j}_k)s_j \\ &= s_i(e_k - s_k)M_{kj}s_j - s_i(e_k - s_k)(M_{jj}\mathbf{j}_j\mathbf{j}_k)s_j \\ &= s_i(e_k - s_k)M_{kj}s_j - ((e_k - s_k)\mathbf{j}_k)s_iM_{jj}\mathbf{j}_j)s_j \\ &= s_ie_kM_{kj}s_j - s_is_kM_{kj}s_j\end{aligned}$$

Where we have noted that $((e_k - s_k)\mathbf{j}_k) = 1 - \sum_i s_i = 0$

□

2.

Consider the fitness matrix for a single population with three alternative strategies:

0	1	2
2	0	2
1	3	0

a. Find the three delta functions algebraically and sketch their nullclines (e.g. $\Delta w_{1-2} = 0$) in barycentric coordinates.

Answer:
The delta functions are

$$\begin{aligned}
\Delta w_{1-2} &= w_1 - w_2 = -2s_1 + s_2 = 0 && \implies 2s_1 - s_2 = 0 \\
\Delta w_{2-3} &= w_2 - w_3 = s_1 - 3s_2 + 2s_3 = 0 && \implies s_1 + 5s_2 = 2 \\
\Delta w_{3-1} &= w_3 - w_1 = s_1 + 2s_2 - 2s_3 = 0 && \implies 3s_1 + 4s_2 = 2
\end{aligned}$$

b. Sector the simplex and indicate possible directions of adjustment for sign-preserving dynamics.

Answer:

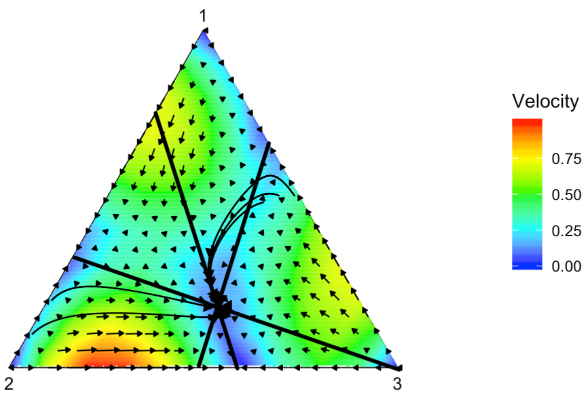

Figure 1: The sectored simplex

Sign-preserving dynamics occurs when all $\dot{s}_i/s_i$ have the same rank-order and are positively correlated with relative fitness; Basins of attraction and equlibria are consistent with replicator dynamics. Rather than considering alternate dynamics, we may consider manipulations of the fitness matrix which preserve rank-order and sign of $\dot{s}_i/s_i$ (*I think this is what the question was asking?*). As shown by question 1, nullifying the diagonals of the fitness matrix by adding constants to each column will preserve relative dynamics and thus is sign-preserving. We may also scale the fitness matrix globally by a real non-zero constant while preserving signed dynamics.

c. Find all steady states and identify them as sinks, sources, saddles, etc from b, and sketch the basins of attraction for the sinks.

Answer:
To find the interior steady state s^*, we solve the equation system from part a to obtain

$$s^* = (\frac{2}{11}, \frac{4}{11}, \frac{5}{11})$$

which is a sink.
The edge equilibria are obtained by solving the 2×2 games on the edges.

- $s_1 = 0$ edge: $(0, \frac{2}{5}, \frac{3}{5})$ is a saddle
- $s_2 = 0$ edge: $(\frac{2}{3}, 0, \frac{1}{3})$ is a saddle
- $s_3 = 0$ edge: $(\frac{1}{3}, \frac{2}{3}, 0)$ is a saddle

The corner equilibria are

- $s_1 = 1$ is a source
- $s_2 = 1$ is a source
- $s_3 = 1$ is a source

d. Which of these steady states are Nash equilibria? Evolutionary equilibria?

Answer:
The interior steady state at $(\frac{2}{11}, \frac{4}{11}, \frac{5}{11})$ is necessarily a mixed Nash equilibrium, where $w_1 = w_2 = w_3$.
Is this point also an Evolutionarily Stable State (Is the Jacobian evaluated at this point negative-definite (all eigenvalues are negative))?

$$J_{ij}(s^*) = \frac{\partial s_i^*(e_k - s_k^*)M_{kj}s_k^*}{\partial s_j^*}$$

which, when computed, is

$$\frac{1}{121}\begin{pmatrix} -54 & -40 & -8 \\ -20 & -124 & -16 \\ -80 & 10 & -130 \end{pmatrix}$$

The eigensystem is

$$\begin{aligned} \lambda_1 &= -14/11 \quad v_1 = (2, 4, 5) \\ \lambda_2 &= -10/11 \quad v_2 = (-1, -2, 3) \\ \lambda_3 &= -4/11 \quad v_3 = (-16, 1, 15) \end{aligned}$$

The first is not parallel to the simplex, while the second two are.
We conclude that the mixed Nash equilibrium at the interior point is an evolutionarily stable point.

e. Assume replicator dynamics. Use eigenvalue or other convenient analytic techniques to check whether each equilibrium has the same stability status (e.g., sink or saddle) as with the sign preserving dynamics.

Answer:
See above for the interior equilibrium.
At each corner, eigenvalues may be negative when largely perpendicular to the simplex, but are positive close to the plane.
At each saddle, eigenvalues parallel to the simplex have one negative and one positive value.