Solutions for Problem Set 3

Ian, Mariana, Hamid

Question 1)

a)

Suppose player one's Pure strategies are {A, B} and player two's pure strategies are {C, D, E}. Therefore, we have a total of 6 action profiles: {(A, C), (A, D), (A, E), (B, C), (B, D), (B, E)}.

b)

Player 1 has two actions for each history and there are 6 strategy profiles form part a, therefore player 1 has 2^6 different pure strategies at stage 2. With the same reason player 2 has 3^6 different pure strategies at stage 2.

c)

We have 6 different action profiles at each history, thus we have 6^3 different action histories after 3 stages of play.

d)

There are $6^3 = 216$ different action histories after stage 3, therefore for player 1 there are 2^{216} different pure strategies and for player 2 there are 3^{216} different pure strategies at stage 4.

Question 2)

An investor (Player 1) spends an amount $k \in [0,20]$ of her capital on a project in a foreign country. Its government (Player 2) then decides on the tax rate $t \in [0,1]$ for the project. The productivity is $r > 0$ so after investment the project is an asset worth $A = (1+r)k$. The government's payoff is $T = tA$ and the investor's payoff is $U = c_0 + c_1$ where $c_0 = 20 - k$ and $c_1 = (1-t)A$.

a) For what values of *t* will a rational investor choose $k > 0$?

The investor will choose $k > 0$ if her final payout is at least 20:

$$U = c_0 + c_1 = (20-k) + (1-t)A = (20-k) + (1-t)(1+r)k \geq 20$$

$$20 - k + k + (r - t - tr)k \geq 20$$

$$(r - t(1+r))k \geq 0 \iff r \geq t(1+r) \iff \boxed{t \leq \frac{r}{1+r}}$$

b) What is the government's best-response to $k > 0$?

The government wishes to maximize $T = tA = t(1+r)k$. For a given $k > 0$:

$$\frac{\partial T}{\partial t} = (1+r)k > 0$$

So the government should tax at the highest possible rate, which is $\boxed{t = 1,}$ or 100%. Of course, this won't encourage much investment from Player 1…

c) Find the SPNE of this static game, and the corresponding payoff vectors

Using backward induction, the government will choose their best response of $t = 1$. This violates the condition from part a), so Player 1 invests $k = 0$. So our Subgame Perfect Nash Equilibrium is $\boxed{(k,t) = (0,1).}$ The payoff vector $\boxed{(U,T) = (20,0)}$

d) Suppose instead that the government commits to *t* before the investor's decision *k*. Find the SPNE of this EFG where the government moves first.

Use backwards induction again, only this time the investor moves last. If $t > \frac{r}{1+r}$, then then Player 1 invests $k = 0$. Otherwise she maximizes *U* with respect to *k*:

$$U = 20 + (r - t - tr)k$$

$$\frac{\partial U}{\partial k} = r - t(1+r) \geq 0, \text{ for } t \leq \frac{r}{1+r}$$

So Player 1 should invest as much as possible, $k = 20$.

Knowing this, the government will tax as high as possible such that Player 1 chooses to invest, which is $t = \frac{r}{1+r}$. So if the government moves first, our SPNE is

$$\boxed{(k,t) = \left(20, \frac{r}{1+r}\right)}$$

e) Find all values of *t* and *k* that maximize the payoff sum in either EFG. Identify a particular such (k^*, t^*) that gives both players higher payoffs than the SPNE outcome in part c.

$$U_{tot} = 20 + (r - t - tr)k + t(1+r)k$$

$$= 20 + rk - tk - trk + tk + trk = 20 + rk$$

This is increasing in *k* and does not depend on t. Because any payout taken from the investor goes to the government, the tax rate does not affect the overall payout. Because it is increasing in *k*, Player 1 should invest as much as possible, $k = 20$. So any $(k, t) = (20, t)$ will maximize total payout.

One possibility that beats the SPNE outcome from part c) is $\boxed{(k^*, t^*) = \left(20, \frac{r}{1+r}\right)}$

f) Write out the NFG in which the investor chooses either $k = 0$ or $k = k^*$, and the government chooses either $t = 1$ or $t = t^*$. Find the NE of this 2x2 game.

	$t = \frac{r}{1+r}$	$t = 1$
$k = 0$	<u>20</u>, <u>0</u>	<u>20</u>,<u>0</u>
$k = 20$	<u>20</u>, 20*r*	0, <u>20(1-*r*)</u>

So by looking at the best responses, the pure NE are $\boxed{\left\{\left(0, \frac{r}{1+r}\right), (0,1)\right\}.}$ Any mixture of these two pure strategies will also be a mixed NE. So our NE is the investor keeping all of their capital, and the government taxing any amount $t \in \left[\frac{r}{1+r}, 1\right]$.

g) Take the game in part f) as the stage game in an infinitely repeated game with discount factor $d < 1$. Generally speaking, what factors determine the discount factor *d*? In this particular situation, which specific factors are especially relevant for *d*?

In this infinitely repeated game the investor has 20 units of capital to invest each period. In general, *d* will depend on the highest available rate of return on capital, and the player's 'patience', or their relative value of having a return now versus later. *d* will also depend on how likely the player believes the game is to continue.

In this particular case, the player's *d* may depend explicitly on *r*. It depends also on the tax rate *t*, and on the player's perception that the government will continue to play, and to behave rationally and tax at rate $t = \frac{r}{1+r}$.

h) For which values of *d* can there be NE grim trigger strategies in the repeated game that sustain an efficient stage game outcome?

Assume that the 'cooperate' option here is the player investing all $k = 20$, and the government taxing at rate $t = \frac{r}{1+r}$. If they are both grim trigger, then by cooperating the player receives:

$$PV(20, 20, \dots, 20, \dots) = \sum_{t=0}^{\infty} d^t 20 = \frac{20}{1-d}$$

If the player at any point 'defects' then they receive:

$$PV(20, 0, 0, \dots, 0, \dots) = 20$$

Cooperation is sustainable if: $\frac{20}{1-d} \geq 20 \iff 1 \geq 1 - d \iff d \geq 0$

So no matter what the player should continue to invest, since they don't gain anything by holding onto their capital.

What about the government? By cooperating they get:

$$PV(20r, 20r, \dots, 20r, \dots) = \sum_{t=0}^{\infty} d^t (20r) = \frac{20r}{1-d}$$

If they defect they get:

$$PV(20(1+r), 0, 0, \dots, 0, \dots) = 20(1+r)$$

So cooperation is sustainable if:

$$\frac{20r}{1-d} \geq 20(1+r) \iff r \geq (1+r)(1-d) = 1 + r - d - dr$$

$$d + dr \geq 1 \iff d \geq \frac{1}{1+r}$$

The government will continue to cooperate if $\boxed{d \geq \frac{1}{1+r}}$.

i) Using your answers in previous parts, predict the circumstances under which the outcome of the foreign investment game is efficient or inefficient.

Based on part h) we will reach the efficient result $(k, t) = \left(20, \frac{r}{1+r}\right)$ so long as the government's *d* is at least $d \geq \frac{1}{1+r}$. If the government is sufficiently patient, it will value gaining the player's trust by taxing at a lower rate, rather than gaining a higher payout by taxing everything for one period. The government is likely to be patient if it expects the game to continue for many additional periods. If it looks like the government might lose power, the player would be wise to think twice before investing their capital

Question 3)

a)

	P	(1-P)
	U	D
U	0,0	4,1
D	1,4	2,2

Game is symmetric, then payoff matrix can be written as:

$$A = \begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix}$$

Let $D(P) \equiv E_\pi(U,P) - E_\pi(D,P)$ be the expected payoff advantage to U. Then:

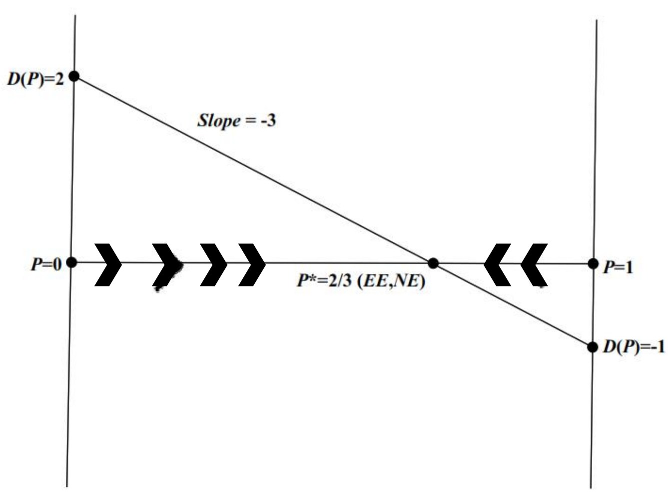
$E_\pi(U,P) = 0P + 4(1-P)$

$E_\pi(D,P) = P + 2(1-P)$

Thus,

$$D(P) = 4(1-P) - P - 2(1-P) = 2 - 3P \;\rightarrow P^* = \frac{2}{3} \; and \; \frac{\partial D(P)}{\partial P} = -3 < 0$$

Therefore, $P^* = \frac{2}{3}$ is the unique evolutionary stable equilibrium.



b)

Now consider different populations:

		P	(1-P)
		U	D
q	U	0,0	4,1
1-q	D	1,4	2,2

$$4(1-q) = q + 2(1-q) \; \rightarrow 2(1-q) = q \; \rightarrow 3q = 3 \; \rightarrow q^* = \frac{2}{3}$$

Now, we have four cases:

If $P < \frac{2}{3} \; \rightarrow \; D(P) \; > 0 \; \rightarrow q \; increases$

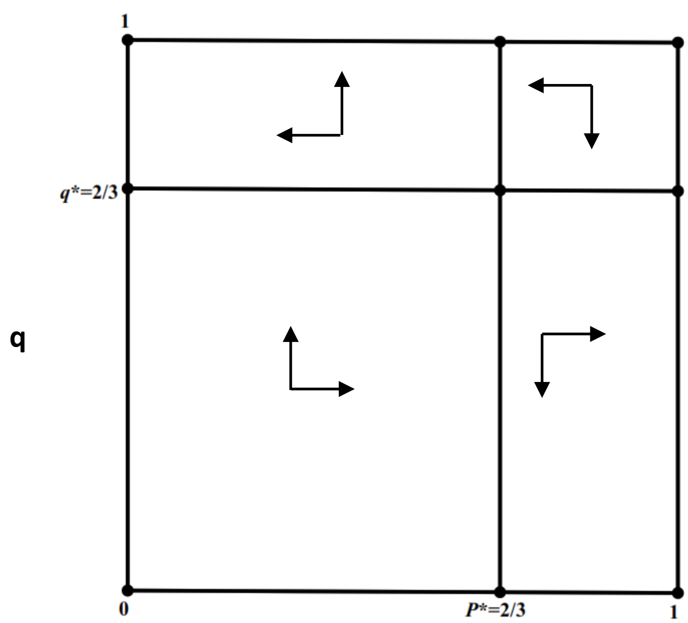
If $P > \frac{2}{3} \; \rightarrow \; D(P) \; > 0 \; \rightarrow q \; decreases$

Since game is symmetric:

If $q < \frac{2}{3} \; \rightarrow \; D(q) \; > 0 \; \rightarrow P \; increases$

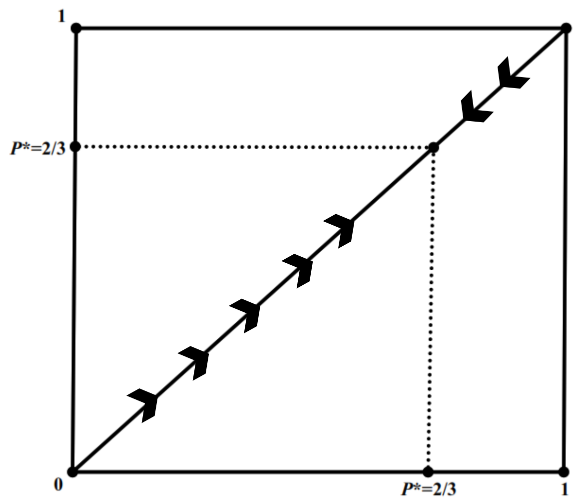
If $q > \frac{2}{3} \rightarrow \; D(q) \; < 0 \; \rightarrow P \; decreases$

From the phase diagram $(P^*, q^*) = \left(\frac{2}{3}, \frac{2}{3}\right)$ is saddle path unstable and (0,1),(1,0) are stable equilibria.



c)

Here p* = $\frac{2}{3}$ is stable. We will be on diagonal because, with only one population we must have p=q. Restricting attention to the diagonal, we see that the equilibrium is stable, as in the diagram below.



d)

	U	D	P	(1-P)
U	0,0	4,1		
D	1,4	2,2		
q				
1-q				

(U,D),(D,U) are pure NE and there is one mixed NE, $(\frac{2}{3}\ U + \frac{1}{3}\ D\ , \frac{2}{3}\ U + \frac{1}{3}\ D\ )$.

If we have one population, then the symmetric mixed NE is the only observable equilibrium. However, for the different case when we have two populations, one of the asymmetric pure NE will be our observable equilibrium. Which one? That depends upon the which basin of attraction we start in.

Question 4 A steel buyer and steel seller form a bilateral monopoly neither has an alternative exchange partner. The seller has constant marginal cost 10 while the buyer has marginal benefit $210-q$ on the q^{th} unit purchased. The two players must first agree on price p. Then the quantity traded is the minimum of what the buyer is willing to buy and what the seller is willing to sell at the given price p.

a. What are the payoffs as functions of p and q?

Let us call π_B the payoff for the buyer and π_S the payoff for the seller.

$\pi_B(p,q) = \int_0^q (210-q-p)dq = 210q - \frac{q^2}{2} - pq$

$\pi_S(p,q) = (p-10)q$

b. Find the Pareto frontier in $(p,\ q)$ space and in payoff space. Identify the point that maximizes the sum of buyer and seller profit.

In order to find the Pareto frontier in (p,q) space, we differentiate π_B and π_S with respect to p and q:

$\frac{\partial \pi_B}{\partial p} = -q \leq 0$

$\frac{\partial \pi_B}{\partial q} = 210 - q - p = 0 \Leftrightarrow p = 210 - q$

$\frac{\partial \pi_S}{\partial p} = q \geq 0$

$\frac{\partial \pi_S}{\partial q} = p - 10 \geq 0 \Leftrightarrow p \geq 10$ *, since the more the seller sells, the better.*

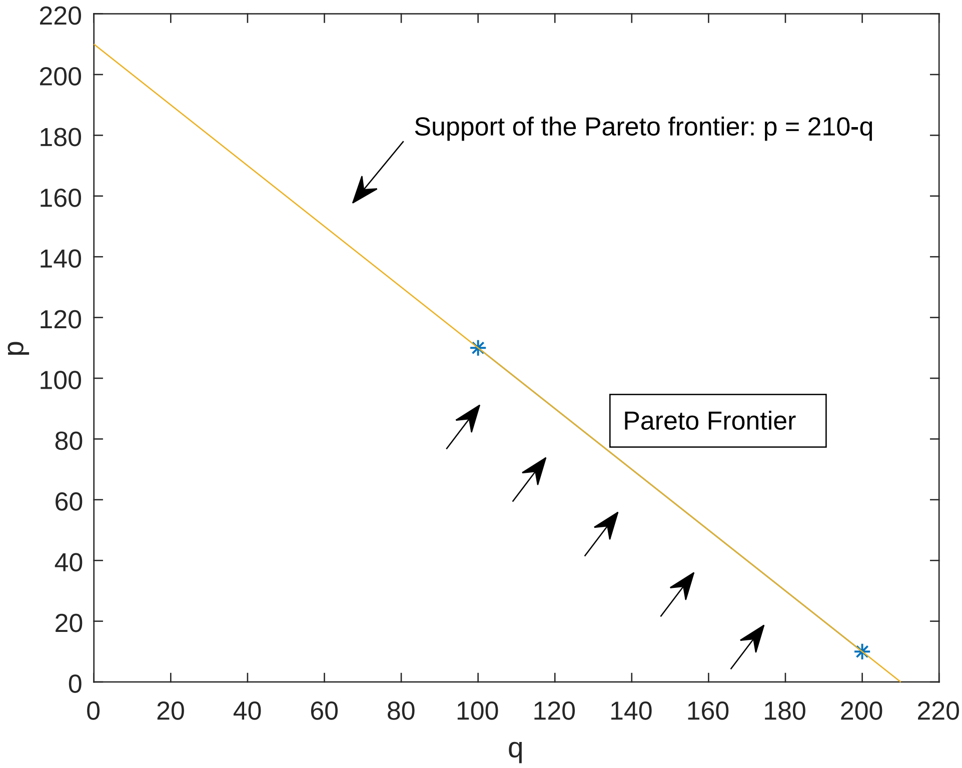
According to the results found in parts c) and d) below, we know that the corresponding Pareto frontier in the (p,q) space is on the locus $p = 210 - q$, delimited by the points $(110, 100)$ and $(10, 200)$. Now, let us plug $p = 210 - q$ into the payoff functions π_B and π_S. Then we can graph the frontier as a parametrized curve, with parameter $q \in [100, 200]$.

$p = 210 - q$
$\pi_B = 210q - \frac{q^2}{2} - pq$
$\pi_S = (p-10)q$

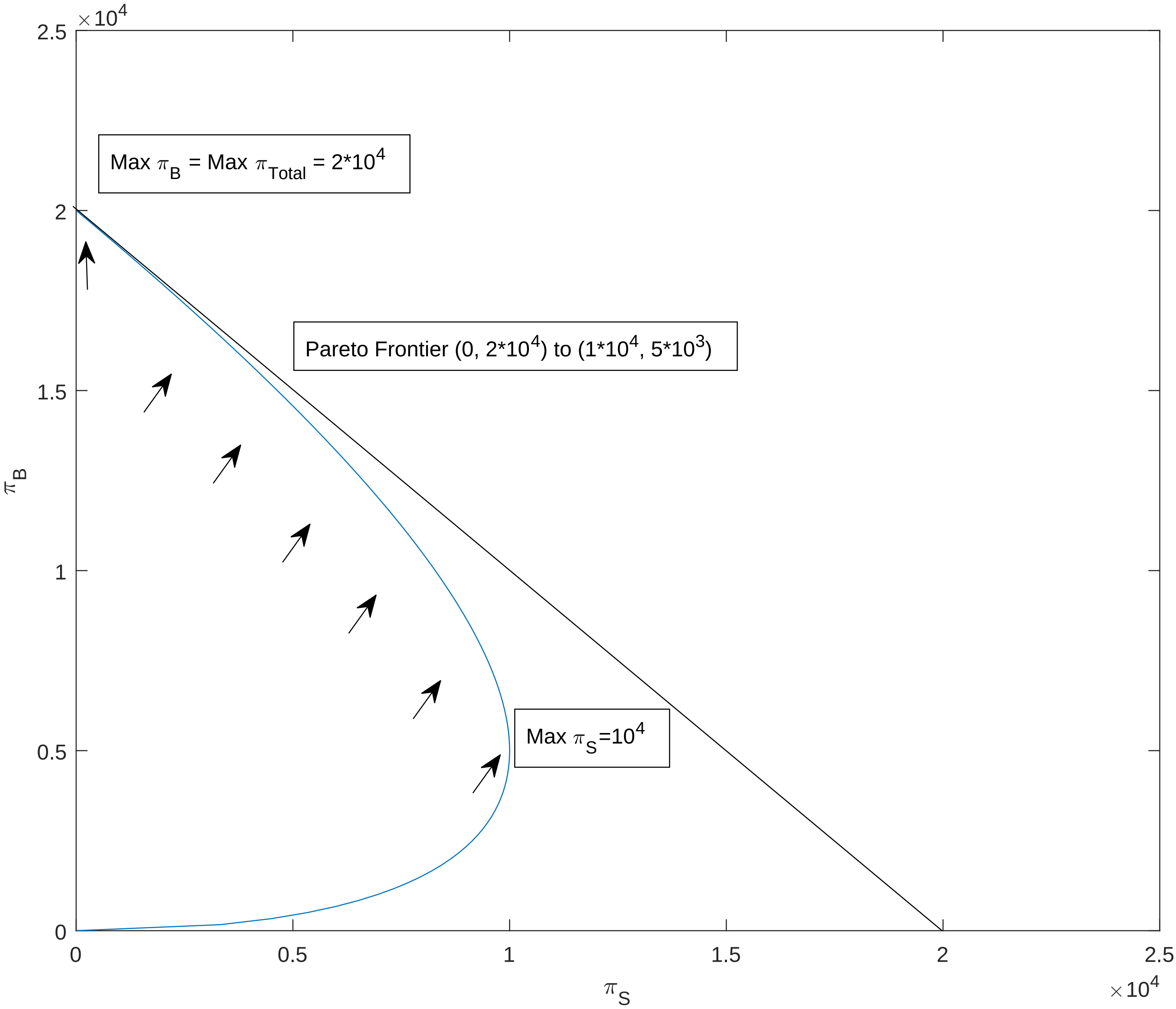
$\pi_B = 210q - \frac{q^2}{2} - (210-q)q$
$\pi_B = \frac{q^2}{2} \Leftrightarrow q = \sqrt{2\pi_B}$

$\pi_S = (210 - q - 10)q$, *since* $100 \leq q \leq 200$
$\pi_S = 200q - q^2$, *now plug* $q = \sqrt{2\pi_B}$ *here to find the Pareto frontier in the* (π_S, π_B) *space:*
$\pi_S = 200\sqrt{2\pi_B} - 2\pi_B$, *since* $5 \times 10^3 \leq \pi_B \leq 2 \times 10^4$

Thus, we can calculate the total payoff in terms of q*:*
$\pi_{Total} = 200q - \frac{q^2}{2}$
F.O.C. w.r.t. q*:*
$200 - q^* = 0 \Leftrightarrow q^* = 200$
Thus, $p^* = 10$*, which corresponds to the point* $({\pi_S}^*, {\pi_B}^*) = (0, 2 \times 10^4)$*, as we can see in the graphics below:*



Support of the Pareto frontier: p = 210-q
Pareto Frontier
p
q
0
20
40
60
80
100
120
140
160
180
200
220



Max π_B = Max π_{Total} = 2*10^4
Pareto Frontier (0, 2*10^4) to (1*10^4, 5*10^3)
Max π_S=10^4
π_B
π_S
$\times 10^4$
0
0.5
1
1.5
2
2.5

c. Suppose the seller chooses price. What is his optimal choice, and what are the resulting profits for buyer and seller?

$\underset{p}{\text{Max}}\, \pi_S$, *but since* $p = 210 - q$, *we have:*

$\underset{p}{\text{Max}}(p - 10)(210 - p)$

F.O.C. w.r.t. p:

$210 - 2p + 10 = 0 \Leftrightarrow p^* = 110$ and $q^* = 100$

$\pi_B(110, 100) = 210 \times 100 - \frac{10^4}{2} - 11 \times 10^3$

$\pi_B(110, 100) = 5 \times 10^3$

$\pi_S(110, 100) = 100 \times 100$

$\pi_S(110, 100) = 10^4$

d. Suppose the buyer chooses price. What is her optimal choice, and what are the resulting profits for buyer and seller?

In letter b), we found that the seller accepts to sell steel only if its price p is at least equal to 10. Then, suppose the buyer pays $p^* = 10$.

$\underset{q}{\text{Max}}\, \pi_B$, *but since* $\pi_B(10, q) = 200q - \frac{q^2}{2}$, *we have:*

$\underset{q}{\text{Max}}(200q - \frac{q^2}{2})$

F.O.C. w.r.t. q:

$200 - q^* = 0 \Leftrightarrow q^* = 200$

Thus, $\pi_B(10, 200) = 210 \times 200 - 2 \times 10^4 - 2 \times 10^3$

$\pi_B(10, 200) = 2 \times 10^4$

and $\pi_S(10, 200) = 0$

e. What is the Nash Bargaining solution to this problem? Assume that the threat points are extreme points of the Pareto frontier. Hints: find the maximum of the product of buyer and seller excess profits, relative to the threat points. Decide whether utility is transferable.

The threat points are :

$\underline{\pi_B} = 5 \times 10^3$, *the minimum payoff for the buyer.*
$\underline{\pi_S} = 0$, *the minimum payoff for the seller.*

For the case of non-transferable utility:

$$\operatorname*{Max}_{\pi_B, \pi_S} (\pi_B - \underline{\pi_B})(\pi_S - \underline{\pi_S}) \;\; s.t. \;\; 200\sqrt{2\pi_B} - 2\pi_B = \pi_S$$

Now plug the constraint above into the maximization and substitute the threat points by their values. Thus, we have the following:

$$\operatorname*{Max}_{\pi_B} (\pi_B - 5 \times 10^3)(200\sqrt{2\pi_B} - 2\pi_B)$$

F.O.C. w.r.t. π_B:
$200\sqrt{2\pi_B} - 2\pi_B + (\pi_B - 5 \times 10^3)(\frac{200\sqrt{2}}{2\sqrt{\pi_B}} - 2) = 0$
$200\sqrt{2\pi_B} - 2\pi_B + 100\sqrt{2\pi_B} - 2\pi_B - \frac{10^6}{\sqrt{2\pi_B}} + 10^4 = 0$
$300\sqrt{2\pi_B} - 4\pi_B - \frac{10^6}{\sqrt{2\pi_B}} + 10^4 = 0$, *do* $\sqrt{2\pi_B} = x$
$300x - 2x^2 - \frac{10^6}{x} + 10^4 = 0$
$-2x^3 + 300x^2 + 10^4 x - 10^6 = 0$
with a little help from Matlab, it holds that:

~~$x_1 = -61.80$~~
$x_2 = 50 \longrightarrow$ ~~$\pi_B = 1,250$~~, *because* π_B *must be at least* 5×10^3
$x_3 = 161.80 \longrightarrow \pi_B = 13,090$ *and* $\pi_S = 6,180.5$

Finally, for the case of transferable utility:
$\operatorname*{Max}_{\pi_B, \pi_S} (\pi_B - \underline{\pi_B})(\pi_S - \underline{\pi_S}) \;\; s.t. \;\; \pi_B + \pi_S = 2 \times 10^4$

Just as before, plug the constraint above into the maximization and substitute the threat points by their values. It holds that:

$$\operatorname*{Max}_{\pi_B} (\pi_B - 5 \times 10^3)(2 \times 10^4 - \pi_B)$$

F.O.C. w.r.t. π_B:
$2 \times 10^4 - 2\pi_B + 5 \times 10^3 = 0$
$\pi_B = 12.5 \times 10^3$ *and* $\pi_S = 7.5 \times 10^3$

$\pi_S + \pi_B(NTU) \leq \pi_S + \pi_B(TU)$

With simple per-unit pricing, utility is not transferable, and the NTU analysis is relevant, as we have seen in the inequality above. The outcome in such bilateral monopoly must be settled through bargaining, for there will be created some positive payoff for each party. Nevertheless, there will not be any transfer of payoff (utility) between them because this is a case of NTU cooperative game.

However, transferable utility can be implemented with more general contracts that allow for lump sum payments between parties without transaction costs, e.g. a flat amount due or payable on delivery, beyond the per-unit price times the quantity sold (sometimes called a two-part tariff).