# Econ 204B: Problem Set 1 Solutions

January 30, 2020

## Part 1: Problems

### Problem 1

$u(a, s) = 1 - (a - s)^2$; $-10 \leq a \leq 20$.
$s$ is a random state variable. Solving (b) first: (b) Optimal action $a^* = \arg\max_a E[1 - (a - s)^2]$; $-10 \leq a \leq$
Since $a$ and $s$ are independent, $E[1-(a-s)^2] = 1-a^2-E(s^2)+2aE(s)$. The Lagrangean for the maximization problem is: $\mathcal{L} = 1-a^2-E(s^2)+2aE(s)-\lambda_1(a-20)-\lambda_2(-a-10)$
The first-order necessary conditions are:
$-2a^* + 2E(s) - \lambda_1 + \lambda_2 = 0, \lambda_1 \geq 0, \lambda_2 \geq 0, a^* \leq 20, a^* \geq -10$
$\lambda_1(a - 20) = 0, \lambda_2(-a - 10) = 0$ (Complementary Slackness Conditions)

The second-order sufficient condition is satisfied; $\frac{\partial^2 \mathcal{L}}{\partial a^2} = -2 < 0$.
$\Rightarrow a^*$ maximizes $u(a, s)$ subject to the given constraints.

Solving the necessary conditions, we arrive at 4 cases:

1. **Case 1** $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. $\Rightarrow a^* = -10$ and $a^* = 20$. This is impossible.
2. **Case 2** $\lambda_1 = \lambda_2 = 0$
   $\Rightarrow 2a^* = 2E(s)$ and $-10 \leq E(s) \leq 20 \Rightarrow a^* = E(s)$ and $-10 \leq a^* \leq 20$
3. **Case 3** $\lambda_1 \neq 0$ and $\lambda_2 = 0 \Rightarrow a^* = 20$ and $-2a^* + 2E(s) = \lambda_1 \geq 0 \Rightarrow E(s) \geq 20$
4. **Case 4** $\lambda_1 = 0$ and $\lambda_2 \neq 0 \Rightarrow a^* = -10$ and $-2a^* + 2E(s) = -\lambda_2 \leq 0 \Rightarrow E(s) \leq -10$

From Cases 1-4, we get:

$$a^* = \begin{cases} -10 & \text{when} E(s) \leq -10 \\ E(s) & \text{when} -10 \leq E(s) \leq 20 \\ 20 & \text{when} E(s) \geq 20 \end{cases}$$

Maximal Payoff= $1 - a^{*2} - E(s^2) + 2a^*E(s)$. Specifically, for Case 2, when $a^* = E(s)$, Maximal Payoff=$1 - [E(s)]^2 - E(s^2) + 2[E(s)]^2 = 1 + [E(s)]^2 - E(s^2) = 1 - var(s)$.

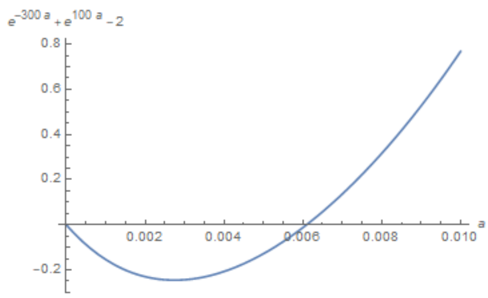
(a) When $s \sim uniform(0,1)$, $E(s) = \frac{1}{2}(0+1) = \frac{1}{2}$, and $var(s) = \frac{1}{12}(1-0)^2 = \frac{1}{12}$
$\Rightarrow E(s^2) = var(s) + [E(s)]^2 = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$
From (b), we know that since $-10 \leq E(s) \leq 20$, $a^* = E(s) = \frac{1}{2}$
Maximal payoff=$1 - {a^*}^2 - E(s^2) + 2a^*E(s) = 1 - \frac{1}{4} - \frac{1}{3} + \frac{1}{2} = \frac{11}{12}$

## Problem 2

(a) Lotteries are defined over the range [\$ 800, \$1200].
CARA utility function: $u(x;a) = 1 - e^{-ax}; a > 0$ Define Lottery L=(800, 1200; $\frac{1}{2}, \frac{1}{2}$). Let $x$ be the certainty equivalent of Lottery L for an agent characterized by a CARA utility function.
By definition, $E_L[u(x_L;a)] = a(x;a)$
$\Rightarrow \frac{1}{2}(1 - e^{-800a}) + \frac{1}{2}(1 - e^{-1200a}) = 1 - e^{-ax}$
$\Rightarrow e^{(x-800)a} + e^{(x-1200)a} - 2 = 0.$
This is a transcendental equation, and it is not particularly easy to get an explicit relationship between $x$ and $a$. Suppose your certainty equivalent is 900. The equation is then modified to:

$$e^{-100a} + e^{300a} - 2 = 0$$

The graph for this function is plotted below:
We can see from the graph that $a \approx 0.006$ solves the equation.

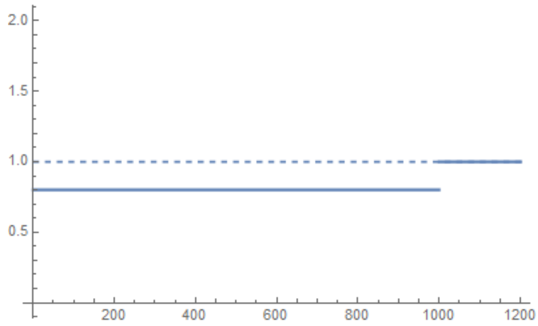



(b) For Lottery L, $\mu_L = \frac{1}{2}(800 + 1200) = 1000$, and $var(L) = \sigma_L^2 = \frac{1}{2}[(800 - 1200)^2 + (1200 - 1000)^2] = 40000$.
$\Rightarrow$ Mean-variance approximation: $\mu_L - c\sigma_L^2 = x \Rightarrow c = \frac{\mu_L - x}{\sigma_L^2} = \frac{1000-x}{40000}$
When $x = 900$, $c = \frac{1}{400}$.

(c) Define Lottery M=(1000, 0; 0.001, 0.999). $\mu_M = 0.001 \times 1000 + 0.999 \times 0 = 1; \sigma_M^2 = 0.001 \times (1000 - 1)^2 + 0.999 \times (0 - 1)^2 = 999$.

$E_M[u(x;a)] = 0.001(1-e^{-1000a}) + 0.999(1-e^0) = 0.001(1-e^{-1000a}) \approx 0.00099$
Mean-Variance Approximation: $\mu_M - c\sigma_M^2 == 1 - \frac{1}{400} \times 999 = -1.4975$

(d) The figure plots the CDF's for Lottery M against receiving 0 with certainty:
We can see that Lottery M first-order stochastically dominates the outcome of




Indicative Plot: Not to Scale

receiving 0 with certainty. This also corroborated by the fact that the expected utility of Lottery M is higher than the utility from receiving 0 with certainty. Denote the lottery that gives 0 with certainty as Lottery O. By FOSD and from the CARA utility function, $M \succ O$. However, according to the mean-variance approximation, $M \prec O$, since the mean-variance approximation of lottery M is negative. This could be due to the fact that the distribution is non-normal.

## Problem 3

(a) $P(d,t_1,t_2) = P(t_1,t_2|d) * P(d) = P(t_1|d) * P(t_2|d) * P(d)$
$P(A,pos,pos) = 0.7 * 0.2 * 0.6 = 0.084$
$P(A,pos,neg) = 0.7 * 0.8 * 0.6 = 0.336$
$P(A,neg,pos) = 0.3 * 0.2 * 0.6 = 0.036$
$P(A,neg,neg) = 0.3 * 0.8 * 0.6 = 0.144$
$P(B,pos,pos) = 0.4 * 0.5 * 0.4 = 0.08$
$P(B,pos,neg) = 0.4 * 0.5 * 0.4 = 0.08$
$P(B,neg,pos) = 0.6 * 0.5 * 0.4 = 0.12$
$P(B,neg,neg) = 0.6 * 0.5 * 0.4 = 0.12$

(b) $P(t_1 = pos, A) = P(t_1 = pos|A) * P(A) = 0.7 * 0.6 = 0.42$
$P(t_1 = pos, B) = P(t_1 = pos|B) * P(B) = 0.4 * 0.4 = 0.16$
$P(t_1 = neg, A) = P(t_1 = neg|A) * P(A) = 0.3 * 0.6 = 0.18$
$P(t_1 = neg, B) = P(t_1 = neg|B) * P(B) = 0.6 * 0.4 = 0.24$

$P(t_2 = pos, A) = P(t_2 = pos|A) * P(A) = 0.2 * 0.6 = 0.12$
$P(t_2 = pos, B) = P(t_2 = pos|B) * P(B) = 0.5 * 0.4 = 0.2$
$P(t_2 = neg, A) = P(t_2 = neg|A) * P(A) = 0.8 * 0.6 = 0.48$
$P(t_2 = neg, B) = P(t_2 = neg|B) * P(B) = 0.5 * 0.4 = 0.2$
$P(A) = 0.6$
$P(B) = 0.4$

(c) $P(A|t) = \frac{P(A,t)}{P(t)} = \frac{P(A,t)}{P(t|A)*P(A)+P(t|B)*P(B)}$

When $t_2$ is not performed,

$P(A|t_1 = pos) = \frac{P(A,t_1=pos)}{P(t_1=pos|A)*P(A)+P(t_1=pos|B)*P(B)} = \frac{0.42}{0.58} = 0.724$

$P(A|t_1 = neg) = \frac{P(A,t_1=neg)}{P(t_1=neg|A)*P(A)+P(t_1=neg|B)*P(B)} = \frac{0.18}{0.42} = 0.429$

When $t_1$ is not performed,

$P(A|t_2 = pos) = \frac{P(A,t_2=pos)}{P(t_2=pos|A)*P(A)+P(t_2=pos|B)*P(B)} = \frac{0.12}{0.32} = 0.375$

$P(A|t_2 = neg) = \frac{P(A,t_2=neg)}{P(t_2=neg|A)*P(A)+P(t_2=neg|B)*P(B)} = \frac{0.48}{0.68} = 0.706$

(d) $P(A|t_1, t_2) = \frac{P(A,t_1,t_2)}{P(t_1,t2)} = \frac{P(A,t_1,t_2)}{P(t_1,t_2|A)*P(A)+P(t_1,t_2|B)*P(B)}$

$= \frac{P(A,t_1,t_2)}{P(t_1|A)*P(t_2|A)*P(A)+P(t_1|B)*P(t_2|B)*P(B)}$

$P(A|pos, pos) = \frac{0.084}{0.7*0.2*0.6+0.4*0.5*0.4} = 0.512$

$P(A|pos, neg) = \frac{0.336}{0.7*0.8*0.6+0.4*0.5*0.4} = 0.808$

$P(A|neg, pos) = \frac{0.036}{0.3*0.2*0.6+0.6*0.5*0.4} = 0.231$

$P(A|neg, neg) = \frac{0.144}{0.3*0.8*0.6+0.6*0.5*0.4} = 0.545$

| | test2 positive | test2 negative |
|---|---|---|
| test1 positive | 0.512 | 0.808 |
| test1 negative | 0.231 | 0.545 |

## Problem 4

(a) Expected payoff without information: 0.1*20+0.5*5+0.4*(-10)=0.5

Expected payoff with perfect information: 0.1*20+0.5*5=4.5

Value of perfect information: 4.5-0.5=4

(b) Likelihood: $(n|\lambda t) = e^{-4\lambda}(\frac{(4\lambda)^n}{n!})$
Great: $P(n|\frac{1}{2}t) = e^{-2}(\frac{2^n}{n!}) = \frac{2^n}{n!e^2}$
Good: $P(n|\frac{1}{4}t) = e^{-1}(\frac{1^n}{n!}) = \frac{1^n}{n!e}$
Poor: $P(n|\frac{1}{8}t) = e^{-\frac{1}{2}}(\frac{(1/2)^n}{n!}) = \frac{(1/2)^n}{n!e^{(1/2)}}$
Posterior Probabilities: $P(\lambda t|n) = \frac{P(n|\lambda t)*P(\lambda t)}{P(n)}$

Great: $P(\frac{1}{2}t|n) = \frac{0.1*\frac{2^n}{n!e^2}}{0.1*\frac{2^n}{n!e^2}+0.5*\frac{1^n}{n!e}+0.4*\frac{(1/2)^n}{n!e^{(1/2)}}}$

Good: $P(\frac{1}{4}t|n) = \frac{0.5*\frac{1^n}{n!e}}{0.1*\frac{2^n}{n!e^2}+0.5*\frac{1^n}{n!e}+0.4*\frac{(1/2)^n}{n!e^{(1/2)}}}$

Poor: $P(\frac{1}{8}t|n) = \frac{0.4*\frac{(1/2)^n}{n!e^{(1/2)}}}{0.1*\frac{2^n}{n!e^2}+0.5*\frac{1^n}{n!e}+0.4*\frac{(1/2)^n}{n!e^{(1/2)}}}$

Expected Utility $= P(\frac{1}{2}t|n) * 20 + P(\frac{1}{4}t|n) * 5 + P(\frac{1}{8}t|n) * (-10)$

EU(0)=-2.8080

EU(1)=0.7462

EU(2)=4.6727

EU(3)=8.6248

...

As shown in the excel spreadsheet attached, as n increases, expected utility from hiring would increase. We will fire the salesman if he sells 0 car in the first week, since in this case only that expected utility is negative. The optimal policy thus is to hire if and only if $n \geq 1$.

From (a), the expected value of uninformed decision is

$$\sum_{x \geq 0} P(n) * EU(n) = 0.5$$

Gross expected value of information is:

$$\sum_{n \geq 1} P(n) * EU(n) - \sum_{n \geq 0} P(n) * EU(n) = -P(0) * EU(0)$$
$$= -0.44009 * (-2.80798) = 1.23575$$

which is the gain from not hiring when $n = 0$.

And net expected value of information is 1.23575-0.04=1.19575

(c) Suppose the salesman sells k cars in the following week

$$\begin{aligned}
P(\lambda t, 2, k) * P(\lambda t) &= P(2|\lambda t) * P(k|\lambda t) * P(\lambda t) \\
&= e^{-4\lambda}(\frac{(4\lambda)^2}{2!}) * e^{-4\lambda}(\frac{(4\lambda)^k}{k!}) * P(\lambda t) \\
P(2, k) &= \sum P(2|\lambda t) * P(k|\lambda t) * P(\lambda t) \\
P(\lambda t|2, k) &= \frac{P(2, k|\lambda t)}{P(2, k)} \\
&= \frac{P(2|\lambda t) * P(k|\lambda t) * P(\lambda t)}{P(2, k)} \\
EU(2, K) &= P(\frac{1}{2}t|2, k) * 20 + P(\frac{1}{4}t|2, k) * 5 + P(\frac{1}{8}t|2, k) * (-10)
\end{aligned}$$

As shown on Spreadsheet, given that the salesman sells 2 cars in the first week, we should not fire him. And that the expected utility of selling two car in the first week and k cars in the second is always positive, we should hire the salesman permanently. Since we will make the same decision of whether to hire of fire the salesman in the second week compare to the first week, value of information for the second week sales is 0. $V_I$(second week sales)=0.

| | **n** | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Great | **P(n\|t/2)** | 0.135335283 | 0.270670566 | 0.270670566 | 0.18044704 | 0.09022352 | 0.03608941 | 0.0120298 | 0.00343709 | 0.00085927 | 0.00019095 | 3.819E-05 |
| Good | **P(n\|t/4)** | 0.367879441 | 0.367879441 | 0.183939721 | 0.06131324 | 0.01532831 | 0.00306566 | 0.00051094 | 7.2992E-05 | 9.124E-06 | 1.0138E-06 | 1.0138E-07 |
| Poor | **P(n\|t/8)** | 0.60653066 | 0.30326533 | 0.075816332 | 0.01263606 | 0.00157951 | 0.00015795 | 1.3163E-05 | 9.4018E-07 | 5.8761E-08 | 3.2645E-09 | 1.6323E-10 |
| Great | **P(t/2)** | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 |
| Good | **P(t/4)** | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| Poor | **P(t/8)** | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 |
| Great | **P(n\|t/2)*P(t/2)** | 0.013533528 | 0.027067057 | 0.027067057 | 0.0180447 | 0.00902235 | 0.00360894 | 0.00120298 | 0.00034371 | 8.5927E-05 | 1.9095E-05 | 3.819E-06 |
| Good | **P(n\|t/4)*P(t/4)** | 0.183939721 | 0.183939721 | 0.09196986 | 0.03065662 | 0.00766416 | 0.00153283 | 0.00025547 | 3.6496E-05 | 4.562E-06 | 5.0689E-07 | 5.0689E-08 |
| Poor | **P(n\|t/8)*P(t/8)** | 0.242612264 | 0.121306132 | 0.030326533 | 0.00505442 | 0.0006318 | 6.318E-05 | 5.265E-06 | 3.7607E-07 | 2.3505E-08 | 1.3058E-09 | 6.529E-11 |
| | **P(n)=sum[P(n\|λt)*P(λt)]** | 0.440085513 | 0.332312909 | 0.14936345 | 0.05375575 | 0.01731831 | 0.00520495 | 0.00146372 | 0.00038058 | 9.0513E-05 | 1.9603E-05 | 3.8697E-06 |
| Great | **P(t/2\|n)** | 0.030752042 | 0.081450512 | 0.181216065 | 0.33567954 | 0.52097186 | 0.69336677 | 0.82186664 | 0.90311635 | 0.94933856 | 0.97407584 | 0.98688435 |
| Good | **P(t/4\|n)** | 0.417963589 | 0.553513618 | 0.615745421 | 0.57029475 | 0.44254636 | 0.29449473 | 0.17453634 | 0.0958955 | 0.05040175 | 0.02585755 | 0.01309878 |
| Poor | **P(t/8\|n)** | 0.551284368 | 0.36503587 | 0.203038514 | 0.09402571 | 0.03648178 | 0.01213849 | 0.00359702 | 0.00098816 | 0.00025968 | 6.6612E-05 | 1.6872E-05 |
| Expected Utility | **EU(n)** | -2.807984888 | 0.746219631 | **4.672663258** | 8.6248075 | 12.2673512 | 15.2184242 | 17.2740442 | 18.5319229 | 19.2361832 | 19.6101384 | 19.8030122 |
| Expected Payoff | **EU(n)*P(n)** | -1.235753469 | 0.247978416 | 0.697925105 | 0.46363297 | 0.21244979 | 0.07921117 | 0.02528431 | 0.00705289 | 0.00174112 | 0.00038442 | 7.6632E-05 |

P(λt, 2, k)=P(2, k|λt)*P(λt)=P(2|λt)*P(k|λt)*P(λt)

| | **k** | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Great | **P(k\|t/2)** | 0.135335283 | 0.270670566 | 0.270670566 | 0.18044704 | 0.09022352 | 0.03608941 | 0.0120298 | 0.00343709 | 0.00085927 | 0.00019095 | 3.819E-05 | |
| Good | **P(k\|t/4)** | 0.367879441 | 0.367879441 | 0.183939721 | 0.06131324 | 0.01532831 | 0.00306566 | 0.00051094 | 7.2992E-05 | 9.124E-06 | 1.0138E-06 | 1.0138E-07 | |
| Poor | **P(k\|t/8)** | 0.60653066 | 0.30326533 | 0.075816332 | 0.01263606 | 0.00157951 | 0.00015795 | 1.3163E-05 | 9.4018E-07 | 5.8761E-08 | 3.2645E-09 | 1.6323E-10 | |
| Great | **P(t/2, 2, k)** | 0.003663128 | 0.007326256 | 0.007326256 | 0.00488417 | 0.00244209 | 0.00097683 | 0.00032561 | 9.3032E-05 | 2.3258E-05 | 5.1684E-06 | 1.0337E-06 | |
| Good | **P(t/4, 2, k)** | 0.033833821 | 0.033833821 | 0.01691691 | 0.00563897 | 0.00140974 | 0.00028195 | 4.6991E-05 | 6.7131E-06 | 8.3913E-07 | 9.3237E-08 | 9.3237E-09 | |
| Poor | **P(t/8, 2, k)** | 0.018393972 | 0.009196986 | 0.002299247 | 0.00038321 | 4.7901E-05 | 4.7901E-06 | 3.9917E-07 | 2.8512E-08 | 1.782E-09 | 9.9002E-11 | 4.9501E-12 | |
| | **P(2,k)=sum[P(λt,2,k)]** | 0.055890921 | 0.050357062 | 0.026542412 | 0.01090635 | 0.00389973 | 0.00126357 | 0.000373 | 9.9773E-05 | 2.4099E-05 | 5.2618E-06 | 1.043E-06 | |
| Great | **P(t/2\|2,k)** | 0.065540659 | 0.145486158 | 0.276020711 | 0.44782821 | 0.62621925 | 0.77307312 | 0.87294814 | 0.93243116 | 0.96510564 | 0.98226149 | 0.99105608 | |
| Good | **P(t/4\|2,k)** | 0.605354509 | 0.671878366 | 0.637353911 | 0.51703558 | 0.36149759 | 0.22313596 | 0.12598169 | 0.06728307 | 0.03482041 | 0.01771969 | 0.00893917 | |
| Poor | **P(t/8\|2,k)** | 0.329104832 | 0.182635475 | 0.086625378 | 0.03513621 | 0.01228315 | 0.00379092 | 0.00107017 | 0.00028577 | 7.3947E-05 | 1.8815E-05 | 4.7459E-06 | |
| Expected Utility | **EU(2,k)** | 1.046537404 | 4.442760245 | 7.840929995 | 11.1903799 | 14.2090415 | 16.5392331 | 18.0781695 | 18.9821808 | 19.4754755 | 19.7336402 | 19.8657701 | |
| Expected Payoff | **EU(2,k)*P(2,k)** | 0.058491939 | 0.223724355 | 0.208117198 | 0.12204618 | 0.05541141 | 0.02089852 | 0.00674319 | 0.00189392 | 0.00046934 | 0.00010383 | 2.072E-05 | 0.6979206 |

## Problem 5

Let $P(newtechnology|production) = P(N|K)$

$P(N|K) = \frac{P(K|N)*P(N)}{P(K|O)*P(O)+P(K|N)*P(N)}$

$= \frac{\frac{1}{\sqrt{2}}*e^{-(1/2)(k-12)^2}*0.1}{\frac{1}{\sqrt{2\pi}}*e^{-(1/2)(k-12)^2}*0.1+\frac{1}{\sqrt{2\pi}}*e^{-(1/2)(k-10)^2}*0.9}$

We want to find k such that $P(N|K) \geq 0.5$ (or $\frac{P(K|N)*P(N)}{P(K|O)*P(O)} \geq 1$)

$\Rightarrow \frac{e^{-(1/2)(k-12)^2}-9e^{-(1/2)(k-10)^2}}{e^{-(1/2)(k-12)^2}+9e^{-(1/2)(k-10)^2}} \geq 0$

$\Rightarrow e^{-\frac{1}{2}(k-12)^2} \geq 9e^{-\frac{1}{2}(k-10)^2}$

$\Rightarrow e^{-\frac{1}{2}(k-12)^2+\frac{1}{2}(k-10)^2} \geq 9$

$\Rightarrow -\frac{1}{2}(k-12)^2 + \frac{1}{2}(k-10)^2 \geq ln(9)$

$\Rightarrow k \geq \frac{Ln9+22}{2} \approx 12.0986$

## Problem 6

The information we have is presented below:

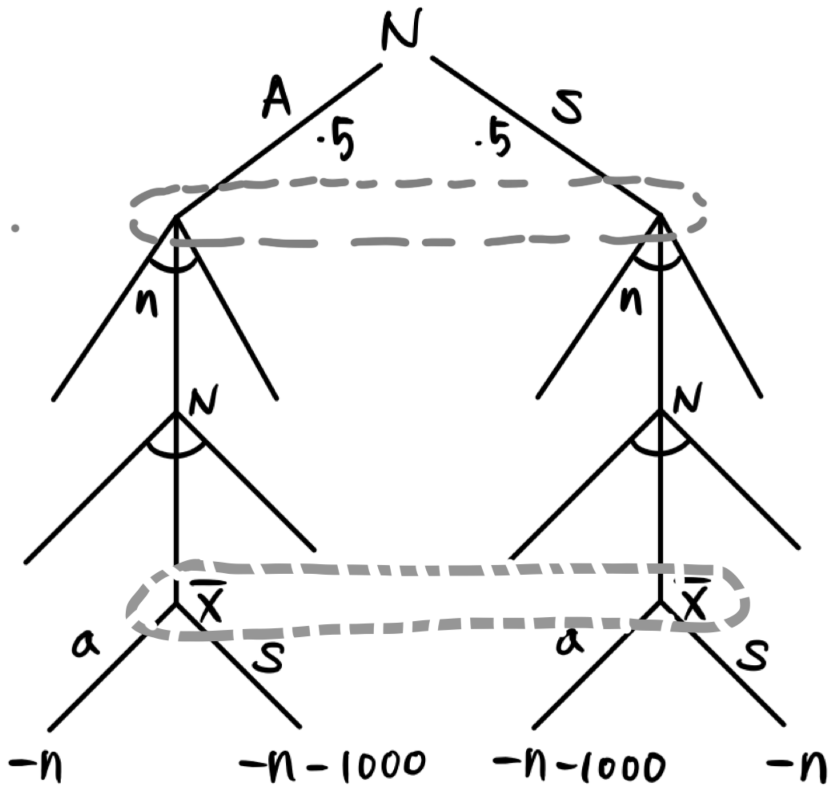
| | Adequate | substandard |
|---|---|---|
| Prior probability | 0.5 | 0.5 |
| Test Results for a product | $\mathcal{N}(1, 9)$ | $\mathcal{N}(-1, 9)$ |
| Utility if correct | 0 | 0 |
| Utility if err | -1000 | -1000 |

(a) The first decision is regarding the choice of the optimal number of units (n) to test. Here, we need to use a new concept, called sufficient statistics. When $x \sim f(x|\theta)$, a function T of x is said to be sufficient if the distribution of x conditional upon T(x) does not depend on $\theta$. Applying Fisher-Neyman factorization lemma, we can say that if $x_1, x_2, ..., x_n \overset{i.i.d}{\sim} \mathcal{N}(\theta, \sigma^2)$, i.e.

$$f(x_1, x_2..x_n|\theta, \sigma^2) = (\frac{1}{\sqrt{2\pi\sigma^2}})^n exp(-\sum_{i=1}^{n} \frac{(x_i-\theta)^2}{2\sigma^2})$$

. In this case, we know that the sufficient statistics for a sample of size $n$ is $(\sum_{i=1}^n x_i, \sum_{i=1}^n x_i^2)$. Furthermore, one-to-one functions of sufficient statistics are also sufficient. Hence, $(\bar{x}, s^2, n)$ is a sufficient statistic for any sample. information. Since the variance is given in this problem, it is sufficient to consider only the sample mean $\bar{x}$ and sample size $n$. We will now show that there is an optimal criterion that depends on $\bar{x}$ independent of $n$.

The decision tree for this problem is presented below:




Decision tree for the problem. Information sets not shown for the two decisions (on n and on a/s) in the current version.

To answer this question, we will use BI method, i.e. first compute both the sample average and SD for a given n to find the decision boundary; then, we will use first order derivative to find the optimal n. For simplicity, we assume that the outcome is the utility minus the cost.

$$E[U(A|X,n)] = -nP(A|\bar{x}) - (n+1000)P(S|\bar{x})$$

$$E[U(S|X,n)] = -nP(S|\bar{x}) - (n+1000)P(A|\bar{x})$$

Choosing adequate(A) is optimal if

$$E[U(A|X,n)] \geq E[U(S|X,n)]$$

i.e.

$$-nP(A|\bar{x}) - (n+1000)P(S|\bar{x}) \geq -nP(S|\bar{x}) - (n+1000)P(A|\bar{x})$$
$$P(A|\bar{x}) \geq P(S|\bar{x})$$

And by Bayes' THM, it is equivalent to $P(\bar{x}|A) \geq P(\bar{x}|S)$. Plugging in the prior probability, we get,

$$exp(-\frac{1}{2}(\frac{\bar{x}-1}{3/\sqrt{n}})^2) \geq exp(-\frac{1}{2}(\frac{\bar{x}+1}{3/\sqrt{n}})^2)$$

$\Rightarrow (\bar{x}+1)^2 \geq (\bar{x}-1)^2$
i.e.

$$\bar{x} \geq 0$$

Therefore, our decision rule is choose A if $\bar{x} \geq 0$ and choose S otherwise.

Secondly, we have to decide how many units to test. The expected outcome is

$$\begin{aligned} E(U|n) =& P(\bar{x} \geq 0)(-nP(A|\bar{x} \geq 0) - (n+1000)P(S|\bar{x} \geq 0)) \\ &+ P(\bar{x} < 0)(-nP(S|\bar{x} < 0) - (n+1000)P(A|\bar{x} < 0) \end{aligned}$$

By Bayes' THM,

$$P(A|\bar{x} \geq 0) = \frac{P(\bar{x} \geq 0|A)P(A)}{P(\bar{x} \geq 0|A)P(A) + P(\bar{x} \geq 0|S)P(S)}$$

$$P(A|\bar{x} < 0) = \frac{P(\bar{x} < 0|A)P(A)}{P(\bar{x} < 0|A)P(A) + P(\bar{x} < 0|S)P(S)}$$

Using the cdf. of standard normal distribution after normalizing the sample,

$$P(\bar{x} \geq 0|A) = P(z \equiv \frac{\bar{x}-1}{3/\sqrt{N}}) = 1 - \Phi(\frac{-1}{3/\sqrt{n}})$$

$$P(\bar{x} < 0|A) = 1 - P(\bar{x} \geq 0|A) = \Phi(\frac{-1}{3/\sqrt{n}})$$

$$P(\bar{x} \geq 0|S) = P(z \equiv \frac{\bar{x}+1}{3/\sqrt{n}}) = 1 - \Phi(\frac{1}{3/\sqrt{n}})$$

$$P(\bar{x} < 0|S) = 1 - P(\bar{x} \geq 0|S) = \Phi(\frac{1}{3/\sqrt{n}})$$

According to the property of standard normal distribution, we know that $P(\bar{x} \geq 0|A) = P(\bar{x} \geq 0|S)$,$P(\bar{x} < 0|A) = P(\bar{x} < 0|S)$, and plug in them into the original equation,

$$E[U|n] = -n - 1000\Phi(\frac{-1}{3/\sqrt{n}})$$

The first order condition is:
$-1 - 1000\phi(\frac{-\sqrt{n*}}{3})(\frac{-1}{6\sqrt{n*}}) = 0$
$\Rightarrow n*^{(\frac{-1}{2})}\frac{1}{2\pi}e^{(\frac{-1}{2}(\frac{\sqrt{n*}}{3})^2)} = 0.006$
Taking natural logs on both sides:
$\frac{-1}{2}ln(n*) - \frac{1}{2}ln(2\pi) - \frac{n*}{12} = ln(0.006)$
Simplifying, we get $n* \approx 42$.

(b) The objective function is the same as before, to minimize expected loss. The only difference here is the prior probabilities. For extra credit, you can solve the problem as follows. Given $\bar{x}$, $P(A) = 0.8$, $P(S) = 0.2$, and costs 1000 and 400, the agent choose $A$ or $S$. As derived in part (a), the optimal decision is adequate $(a)$ if

$$\begin{aligned} \frac{P(\bar{X}|S)P(S)}{400} &\leq \frac{P(\bar{X}|A)P(A)}{1000} \\ \Leftrightarrow \frac{P(\bar{X}|A)}{P(\bar{X}|S)} &\geq \tfrac{5}{8} \\ \bar{X} \geq -\frac{1}{2}(\ln 8 - \ln 5)\frac{9}{n} &\equiv c(n) \end{aligned}$$

$c(n)$ replace 0 in (a) and represents the indifferent point between choosing $A$ and choosing $S$. The expected utility under this optimal strategy given $n$ is

$$\begin{aligned} E[u|n] = & P(\bar{X} \geq c(n)) \left(-nP(A|\bar{X} \geq c(n)) - (n+1000)P(S|\bar{X} \geq c(n))\right) \\ & + P(\bar{X} < c(n)) \left(-nP(S|\bar{X} < c(n)) - (n+400)P(A|\bar{X} < c(n))\right) \end{aligned}$$

By Bayes' theorem,

$$\begin{aligned} P(A|\bar{X} \geq c(n)) &= \frac{P(A|\bar{X} \geq c(n)|A)P(A)}{P(A|\bar{X} \geq c(n)|A)P(A) + P(A|\bar{X} \geq c(n)|S)P(S)} \\ P(A|\bar{X} < c(n)) &= \frac{P(A|\bar{X} < c(n)|A)P(A)}{P(A|\bar{X} < c(n)|A)P(A) + P(A|\bar{X} < c(n)|S)P(S)} \end{aligned}$$

For the ease of derivation, express the probability using cdf of standardized normal distribution.

$$P(\bar{X} \geq c(n)|A) = P\left(z \equiv \frac{\bar{X} - 1}{3/\sqrt{n}} \geq \frac{c(n) - 1}{3/\sqrt{n}}\right) = 1 - \Phi\left(\frac{c(n) - 1}{3/\sqrt{n}}\right)$$

Similarly,

$$\begin{aligned} P(\bar{X} < c(n)|A) &= \Phi\left(\frac{c(n) - 1}{3/\sqrt{n}}\right) \\ P(\bar{X} \geq c(n)|S) &= 1 - \Phi\left(\frac{c(n) + 1}{3/\sqrt{n}}\right) \\ P(\bar{X} < c(n)|S) &= \Phi\left(\frac{c(n) + 1}{3/\sqrt{n}}\right) \end{aligned}$$

Also, using the fact that

$$P(\bar{X} \geq c(n), A) + P(\bar{X} < c(n), A) + P(\bar{X} \geq c(n), S) + P(\bar{X} < c(n), S) = 1$$

The final expression of expected utility under the optimal strategy given $n$ is

$$E[u|n] = -n - 1000\left(1 - \Phi\left(\frac{c(n)+1}{3/\sqrt{n}}\right)\right) \cdot \frac{1}{5} - 400\Phi\left(\frac{c(n)-1}{3/\sqrt{n}}\right) \cdot \frac{4}{5}$$

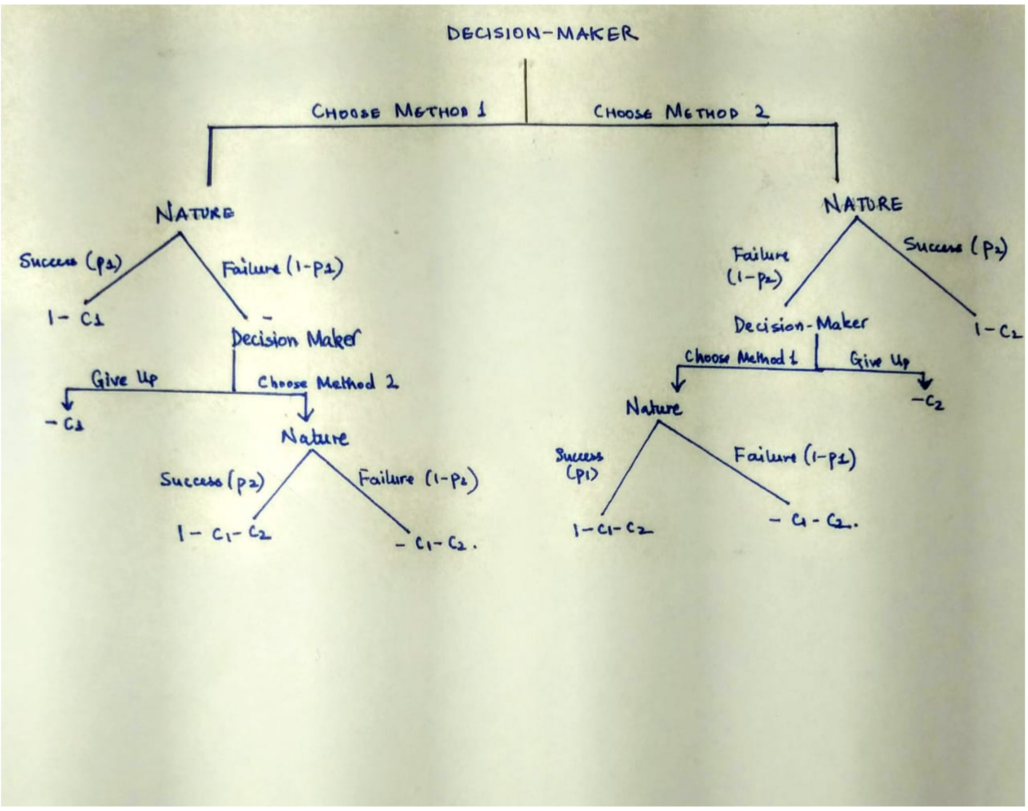
remind that $c(n) = -\frac{1}{2}(\ln 8 - \ln 5)\frac{9}{n}$, similar to (a), the optimal $n$ is the value which maximizes this expected utility.

$$n^* = 32$$

(c) This is a dynamic decision theoretic problem, somewhat beyond the scope of this course. See DeGroot (2005), Optimal Statistical Decisions.

## Problem 7

For any method $i \in 1, 2, ...K$, the expected payoff from employing method $i$ is $p_i(1 - c_i) + (1 - p_i)(-c_i) = p_i - c_i$.
Let Method 1 and Method 2. characterized by $(p_1, c_1)$ and $(p_2, c_2)$ respectively, be any two methods for which expected payoff is positive, i.e, $p_i - c_i \geq 0$ for $i \in 1, 2$. The decision tree looks as follows:




Decision tree for k=2

Solving through backwards induction,
**At node 2**:
Expected payoff from continuing with method 2=$p_2(1-c_1-c_2)+(1-p_2)(-c_1-c_2)=p_2-c_1-c_2$.
Expected payoff from stopping after method 1=$-c1$.
Decision: Choose to continue with Method 2 iff $p_2-c_1-c_2 \geq -c_1 \Rightarrow p_2-c_2 \geq 0$. Stop if $p_2-c_2<0$.

**At node 1**:
Expected payoff from choosing method 1=$p_1(1-c_1)+(1-p_1)(-c_1+p_2-c_2)=(p_1-c_1)+(1-p_1)(p_2-c_2)$
Expected payoff from choosing method 1=$p_2(1-c_2)+(1-p_2)(-c_2+p_1-c_1)=(p_2-c_2)+(1-p_2)(p_1-c_1)$
Decision: Choose Method 1 if $(p_1-c_1)+(1-p_1)(p_2-c_2) \geq (p_2-c_2)+(1-p_2)(p_1-c_1) \Rightarrow p_1c_2 \geq p_2c_1$
$\Rightarrow \frac{p_1}{c_1} \geq \frac{p_2}{c_2}$, and $p_i-c_i \geq 0$ for $i \in 1,2$.

For $N>2$, a sketch of a proof is presented below. We claim here that the optimal order of decision is to try methods characterized by $(p_i,c_i)$ for $i \in 1,2...K, K>2$, in decreasing order of $\frac{p_i}{c_i}$. To see this, assume, by way of contradiction, that there exists another optimal order wherein the methods are not arranged by descending order of $\frac{p_i}{c_i}$. This implies that there exists at least one pair of methods $(p_j,cj)$ and $(p_k,c_k)$, such that $\frac{p_j}{c_j} \geq \frac{p_k}{c_k}$, but $(p_j,c_j)$ is tried after $(p_k,c_k)$. However, this contradicts the result obtained for the $K=2$ case.

## Part 2: Textbook Problems

### 6.C.18

Bernoulli utility function $U(x)=\sqrt{x}$, this means the player is risk aversion.

1. Arrow-Pratt Coefficients (We assume U(x)=U(w))
   CARA: $A(w)=-\frac{U''(w)}{U'(w)}$, at $w=5$, $A(5)=\frac{0.25*x^{-1.5}}{0.5x^{-0.5}}=\frac{1}{10}$
   CRRA: $R(w)=-w\frac{U''(w)}{U'(w)}$, at $w=5$, $R(5)=5*\frac{0.25*x^{-1.5}}{0.5x^{-0.5}}=\frac{1}{2}$
2. Certainty equivalent and probability premium for a gamble (16,4;0.5,0.5). Compared to question b, both wealth outcomes are higher. But the ratio between the highest wealth to the lowest wealth is becoming lower in case c.

$$CE_b \equiv U^{-1}[E(U)]=(0.5*U(16)+0.5*U(4))^2=9$$

$$\begin{aligned} PP_b &\equiv \frac{U(E(x)-E(U(x))}{U(x_{large})-U(x_{small})} \\ &= \frac{U(0.5*16+0.5*4)-E[0.5U(16)+0.5U(4)]}{U(16)-U(4)} \approx 0.08 \end{aligned}$$

3. Certainty equivalent and probability premium for a gamble (36,16;0.5,0.5)

$$CE_c \equiv U^{-1}[E(U)] = (0.5 * U(36) + 0.5 * U(16))^2 = 25$$

$$\begin{aligned} PP_c &\equiv \frac{U(E(x) - E(U(x))}{U(x_{large}) - U(x_{small})} \\ &= \frac{U(0.5 * 16 + 0.5 * 4) - E[0.5U(36) + 0.5U(16)]}{U(36) - U(16)} \approx 0.05 \end{aligned}$$

From the calculation, we can see that $CE_b < CE_c$ but $PP_b > PP_c$. Higher CE value means less risk averse when facing higher E[wealth], and it is consistent with the result $A(x) = \frac{1}{2x}$. This can be explained because the expected wealth is higher in questionc.

Lower PP means lower excess in wining probability over fair odds, which makes the individual indifferent between the certain outcome x and a gamble between the two outcomes $x + \epsilon$ and $x - \epsilon$. This is also true because the ratio between the highest wealth to the lowest wealth is becoming lower in case c, which reduced the cancavity of the utility function(in the interval $[16, 36]$). And it is consistent with the above result because lower concavity means less risk aversion facing the increase of the wealth bundle.

**6.C.20**

*Proof:*

$$U(CE) = E(U) = \frac{1}{2}U(x + \epsilon) + U(x - \epsilon)$$

F.O.D.

$$U'(CE)\frac{\partial CE}{\partial \epsilon} = \frac{1}{2}U'(x + \epsilon) - \frac{1}{2}U'(x - \epsilon)$$

$$\lim_{\epsilon \to 0} U'(CE) = U'(x)$$

S.O.D.

$$U''(CE)(\frac{\partial CE}{\partial \epsilon})^2 + U'(CE)\frac{\partial^2 CE}{\partial \epsilon^2} = \frac{1}{2}U''(x + \epsilon) + \frac{1}{2}U''(x - \epsilon)$$

$$\frac{\partial^2 CE}{\partial \epsilon^2} = [\frac{1}{2}U''(x + \epsilon) + \frac{1}{2}U''(x - \epsilon) - U''(CE)(\frac{\frac{1}{2}U'(x + \epsilon) - \frac{1}{2}U'(x - \epsilon)}{U'(CE)})^2]\frac{1}{U'(CE)}$$

,if $U'(CE) \neq 0$

Taking the limit as $\epsilon \to 0$, we have,

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{\partial^2 CE}{\partial \epsilon^2} &= \lim_{\epsilon \to 0}[\frac{1}{2}U''(x) + \frac{1}{2}U''(x) - U''(CE)((\frac{\frac{1}{2}U'(x) - \frac{1}{2}U'(x)}{U'(CE)})^2]\frac{1}{U'(CE)} \\ &= \lim_{\epsilon \to 0} \frac{U''(x)}{U'(CE)} \\ &= \frac{U''(x)}{U'(x)} = -r_A(x) \end{aligned}$$