An $n$-player game $\Gamma$ in strategic form is a $2^n$-tuple $\Gamma = \langle S^1, \ldots, S^n, \pi^1, \ldots, \pi^n \rangle$ where $S^i$ is player $i$'s set of pure strategies and $\pi^i : \prod_{j=1}^{n} S^j \to \mathbb{R}$ is $i$'s payoff function. If each $S^i$ is a finite set, then $\Gamma$ is called a finite game. A mixed strategy for player $i$ is a probability measure on $i$'s set of pure strategies. So, player $i$'s set of mixed strategies is

$$\Delta(S^i) = \left\{ \sigma^i : S^i \to \mathbb{R}_+ \mid \sum_{s^i \in S^i} \sigma(s^i) = 1 \right\}$$

where $\sigma^i(s^i)$ is the probability with which $i$ chooses the pure strategy $s^i \in S^i$. If the players choose the $n$-tuple of mixed strategies $\sigma = (\sigma^1, \ldots, \sigma^n)$, then the expected payoff to player $i$ is

$$v^i(\sigma) = \sum_{(s^1, \ldots, s^n) \in S^1 \times \cdots \times S^n} \prod_{j=1}^{n} \sigma^j(s^j) \pi^i(s^1, \ldots, s^n).$$

The following notation will be helpful:

$$\sigma^{-i} = (\sigma^1, \ldots, \sigma^{i-1}, \sigma^{i+1}, \ldots, \sigma^n),$$

$$\left(\sigma^{-i}, \tilde{\sigma}^i\right) = (\sigma^1, \ldots, \sigma^{i-1}, \tilde{\sigma}^i, \sigma^{i+1}, \ldots, \sigma^n).$$

**Definition 1** An $n$-tuple of mixed strategies $\sigma = (\sigma^1, \ldots, \sigma^n)$ is a **Nash equilibrium** if for every $i$ it is true that $v^i(\sigma) \geq v^i(\sigma^{-i}, \tilde{\sigma}^i)$ for every $\tilde{\sigma}^i \in \Delta(S^i)$.

**Theorem 1 (Nash)** Every finite game possesses a Nash equilibrium.

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*With the assistance of Amanda Friedenberg. nashproof-01-04-07*
Nash’s theorem will be proved as an easy corollary of a more general existence theorem.

**Theorem 2** Consider an $n$-player game $\Gamma = (D^1, \ldots, D^n, v^1, \ldots, v^n)$ where $D^i$ is the set of pure strategies available to player $i$ and $v^i : \prod_{j=1}^{n} D^j \rightarrow \mathbb{R}$ is $i$’s payoff function. If each $D^i$ is a compact and convex subset of Euclidean space, and each $v^i$ is quasiconcave in $d^i$ and continuous, then $\Gamma$ has a Nash equilibrium in pure strategies.

(To say that $v^i$ is quasiconcave in $d^i$ means that

$$v^i(d^{-i}, \alpha d^i + (1-\alpha) \tilde{d}^i) \geq \min \left\{ v^i(d^{-i}, d^i), v^i(d^{-i}, \tilde{d}^i) \right\}$$

for every $\alpha \in [0, 1].$) Theorem 2 is in turn an immediate consequence of Kakutani’s Fixed Point Theorem. Before stating Kakutani’s theorem, a definition will be useful.

**Definition 2** A correspondence $\phi$ from a subset $T$ of Euclidean space to a compact subset $V$ of Euclidean space is upper hemicontinuous at a point $x \in T$ if $x_r \rightarrow x$, $y_r \rightarrow y$, where $y_r \in \phi(x_r)$ for every $r$, implies $y \in \phi(x)$. The correspondence $\phi$ is upper hemicontinuous if it is upper hemicontinuous at every $x \in T$.

**Theorem 3 (Kakutani)** If $T$ is a nonempty compact and convex subset of Euclidean space, and $\phi$ is an upper hemicontinuous, nonempty, and convex-valued correspondence from $T$ to $T$, then $\phi$ has a fixed point, that is, there is an $x \in T$ such that $x \in \phi(x)$.

**Proof of Theorem 2.** For each $i$ define a best reply correspondence $\phi^i$ from $\prod_{j=1}^{n} D^j$ to $D^i$ as follows. For any $d \in \prod_{j=1}^{n} D^j$, let

$$\phi^i(d) = \left\{ \tilde{d}^i \in D^i \mid v^i(d^{-i}, \tilde{d}^i) \geq v^i(d^{-i}, d^i) \right\}.$$  

The set $\phi^i(d)$ is the set of strategies that maximizes $i$’s payoff given the strategies of the other players prescribed by $d$; it is nonempty since $D^i$ is compact and $v^i$ is continuous. The correspondence $\phi^i(d)$ is convex-valued since $v^i$ is quasiconcave in $d^i$. To see that $\phi^i$ is upper hemicontinuous, consider a sequence $d_r$ in $\prod_{j=1}^{n} D^j$ converging to $d$, and a sequence $\tilde{d}_r$ in $D^i$ converging to $\tilde{d}^i$, where $\tilde{d}_r \in \phi^i(d_r)$
for every \( r \). For any \( \tilde{d}_i \in D_i \), we have \( v^i(d_i^{-i}, \tilde{d}_i) \geq v^i(d_i^{-i}, d_i) \). Therefore \( v^i(d_i^{-i}, \tilde{d}_i) \geq v^i(d_i^{-i}, d_i) \), since \( v^i \) is continuous, i.e. \( \tilde{d}_i \in \phi(d) \). This shows that \( \phi \) is upper hemicontinuous. Now define a correspondence \( \phi \) from \( \prod_{i=1}^n D_i \) to

\[
\prod_{i=1}^n D_i \text{ by}
\]

\[
\phi(d) = \phi^1(d) \times \cdots \times \phi^n(d).
\]

The set \( \prod_{i=1}^n D_i \) is a compact and convex subset of Euclidean space since each \( D_i \) is. The correspondence \( \phi \) is upper hemicontinuous, nonempty, and convex-valued since each \( \phi^i \) is. And it is easy to see that a fixed point of \( \phi \) is just a Nash equilibrium of \( \Gamma \). ■

Finally, we have to say why Theorem 1 follows from Theorem 2. To see this, just let \( D_i = \Delta(S_i) \). Each \( D_i \) is then a compact and convex subset of a Euclidean space, and each \( v^i \) is quasiconcave in \( d_i \) (in fact, linear in each variable) and continuous. So, by Theorem 2, there is a Nash equilibrium of the game in which each player \( i \) chooses a pure strategy from \( \Delta(S_i) \). But this is, of course, just a Nash equilibrium in mixed strategies of the original game.