Nash Equilibrium: Existence

Adam Brandenburger*

Version 01/04/07

An n-player game Γ in strategic form is a $2n$-tuple $\Gamma = \left\langle S^1, \ldots, S^n, \pi^1, \ldots, \pi^n \right\rangle$ where S^i is player i's set of pure strategies and $\pi^i : \prod_{j=1}^{n} S^j \to \Re$ is i's payoff function. If each S^i is a finite set, then Γ is called a finite game. A mixed strategy for player i is a probability measure on i's set of pure strategies. So, player i's set of mixed strategies is

$$\Delta(S^i) = \left\{ \sigma^i : S^i \to \Re_+ \mid \sum_{s^i \in S^i} \sigma(s^i) = 1 \right\},$$

where $\sigma^i(s^i)$ is the probability with which i chooses the pure strategy $s^i \in S^i$. If the players choose the n-tuple of mixed strategies $\sigma = \left(\sigma^1, ..., \sigma^n\right)$, then the expected payoff to player i is

$$v^i(\sigma) = \sum_{(s^1,\ldots,s^n) \in S^1 \times \cdots \times S^n} \prod_{j=1}^{n} \sigma^j(s^j) \pi^i(s^1, \ldots, s^n).$$

The following notation will be helpful:

$$\begin{aligned} \sigma^{-i} &= (\sigma^1, \ldots, \sigma^{i-1}, \sigma^{i+1}, \ldots, \sigma^n), \\ (\sigma^{-i}, \widetilde{\sigma}^i) &= (\sigma^1, \ldots, \sigma^{i-1}, \widetilde{\sigma}^i, \sigma^{i+1}, \ldots, \sigma^n). \end{aligned}$$

Definition 1 *An n-tuple of mixed strategies $\sigma = (\sigma^1, \ldots, \sigma^n)$ is a **Nash equilibrium** if for every i it is true that $v^i(\sigma) \geq v^i(\sigma^{-i}, \widetilde{\sigma}^i)$ for every $\widetilde{\sigma}^i \in \Delta(S^i)$.*

Theorem 1 (Nash) *Every finite game possesses a Nash equilibrium.*

*With the assistance of Amanda Friedenberg. nashproof-01-04-07

Nash's theorem will be proved as an easy corollary of a more general existence theorem.

Theorem 2 *Consider an n-player game* $\Gamma = \langle D^1, \ldots, D^n, v^1, \ldots, v^n \rangle$ *where* D^i *is the set of pure strategies available to player i and* $v^i : \prod_{j=1}^{n} D^j \to \Re$ *is i's payoff function. If each* D^i *is a compact and convex subset of Euclidean space, and each* v^i *is quasiconcave in* d^i *and continuous, then* Γ *has a Nash equilibrium in pure strategies.*

(To say that v^i is quasiconcave in d^i means that

$$v^i(d^{-i}, \alpha d^i + (1-\alpha)\widetilde{d^i}) \geq \min\left\{v^i(d^{-i}, d^i), v^i(d^{-i}, \widetilde{d^i})\right\}$$

for every $\alpha \in [0, 1]$.) Theorem 2 is in turn an immediate consequence of Kakutani's Fixed Point Theorem. Before stating Kakutani's theorem, a definition will be useful.

Definition 2 *A correspondence* ϕ *from a subset* T *of Euclidean space to a compact subset* V *of Euclidean space is* ***upper hemicontinuous at a point*** $x \in T$ *if* $x_r \to x$, $y_r \to y$, *where* $y_r \in \phi(x_r)$ *for every* r, *implies* $y \in \phi(x)$. *The correspondence* ϕ *is* ***upper hemicontinuous*** *if it is upper hemicontinuous at every* $x \in T$.

Theorem 3 (Kakutani) *If* T *is a nonempty compact and convex subset of Euclidean space, and* ϕ *is an upper hemicontinuous, nonempty, and convex-valued correspondence from* T *to* T, *then* ϕ *has a fixed point, that is, there is an* $x \in T$ *such that* $x \in \phi(x)$.

Proof of Theorem 2. For each i define a *best reply* correspondence ϕ^i from $\prod_{j=1}^{n} D^j$ to D^i as follows. For any $d \in \prod_{j=1}^{n} D^j$, let

$$\phi^i(d) = \left\{\widehat{d^i} \in D^i \mid v^i(d^{-i}, \widehat{d^i}) \geq v^i(d^{-i}, \widetilde{d^i}) \text{ for every } \widetilde{d^i} \in D^i\right\}.$$

The set $\phi^i(d)$ is the set of strategies that maximizes i's payoff given the strategies of the other players prescribed by d; it is nonempty since D^i is compact and v^i is continuous. The correspondence $\phi^i(d)$ is convex-valued since v^i is quasiconcave in d^i. To see that ϕ^i is upper hemicontinuous, consider a sequence d_r in $\prod_{j=1}^{n} D^j$ converging to d, and a sequence $\widehat{d^i_r}$ in D^i converging to $\widehat{d^i}$, where $\widehat{d^i_r} \in \phi^i(d_r)$

for every r. For any $\widetilde{d}_i \in D^i$, we have $v^i(d_r^{-i}, \widehat{d_r^i}) \geq v^i(d_r^{-i}, \widetilde{d^i})$. Therefore $v^i(d^{-i}, \widehat{d^i}) \geq v^i(d^{-i}, \widetilde{d^i})$, since v^i is continuous, i.e. $\widehat{d^i} \in \phi(d)$. This shows that ϕ^i is upper hemicontinuous. Now define a correspondence ϕ from $\prod_{i=1}^{n} D^i$ to $\prod_{i=1}^{n} D^i$ by

$$\phi(d) = \phi^1(d) \times \cdots \times \phi^n(d).$$

The set $\prod_{i=1}^{n} D^i$ is a compact and convex subset of Euclidean space since each D^i is. The correspondence ϕ is upper hemicontinuous, nonempty, and convex-valued since each ϕ^i is. And it is easy to see that a fixed point of ϕ is just a Nash equilibrium of Γ. ■

Finally, we have to say why Theorem 1 follows from Theorem 2. To see this, just let $D^i = \Delta(S^i)$. Each D^i is then a compact and convex subset of a Euclidean space, and each v^i is quasiconcave in d^i (in fact, linear in each variable) and continuous. So, by Theorem 2, there is a Nash equilibrium of the game in which each player i chooses a pure strategy from $\Delta(S^i)$. But this is, of course, just a Nash equilibrium in mixed strategies of the original game.