ECON204B PS 3 – Answer Key

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Problem #1

Assuming game is NFG,

a)Let's assume Player 1 has strategies (X, Y) and Player 2 has strategies (V, W, Z), then from 2*3=6, we have 6 action profiles for first stage : $(X, V), (X, W), (X, Z), (Y, V), (Y, W), (Y, Z)$

b) At stage 2, if we consider about NFG (We also only consider about NFG in the following questions for Problem 1), Player 1 will have 2^6 pure strategies and Player 2 will have 3^6 pure strategies.

c)We have three stages with 6 different action profiles, so after stage 3 we will have $6^3 = 216$ different action histories.

d)We already know that after stage three, we have 216 action histories. So, for Player 1, at stage 4, we will have 2^{216} pure strategies and for Player 2 we will have 3^{216} pure strategies.

Problem #2

a) $k > 0$ is a BR only if the after-tax gain on investment exceeds the reduction k in current consumption:
$(1-t)(1+r)k \geq k$
$(1-t)(1+r) \geq 1$
so $t \leq \frac{r}{r+1}$,that is, $t \in [0, \frac{r}{r+1}]$ is necessary for $k \geq 0$.
Indeed the unique BR to $t \in [0, \frac{r}{r+1})$ is $k = 20$; while BR to $t = \frac{r}{r+1}$ is any $k \in [0, 20]$.

b) Government payoff: $T = tA$

$\pi_g = tA = t(1+r)k$
The FOC

$$\frac{\partial \pi_g}{\partial t} = A = (1+r)k > 0$$

$\implies t = 1$ is the unique BR when $k > 0$.

c) If Investor chooses k first, then by BI, if $k > 0$, Government would choose $t = 1$. Thus, Investor would choose $k = 0$. The SPNE payoff vector would be (20,0).

d) If Government moves first, then by BI, if $t > \frac{r}{r+a}$, Investor would choose $k = 0$. Thus, Government would choose $t < \frac{r}{r+1}$ but very close to $\frac{r}{r+1}$. Then Investor would choose $k = 20$ as unique BR.

e) Sum of utility:

$$\begin{aligned}\pi_I + \pi_g &= 20 - k + (1-t)A + tA \\ &= 20 - k + A \\ &= 20 + rk\end{aligned}$$

The payoff sum is only related to k. When $k = max(k) = 20$, $\pi_I + \pi_g$ reaches maximum value.
For both players to get higher payoffs than at the bad SPNE,
g chooses $t^\star < \frac{r}{r+1}$, say $t^\star = 0.5$ (assuming that $r > 1$).
In this case, the BR is $k^\star = 20$ and payoff sum is maximized.

f) The NFG is shown below:

	$k = 0$	$k = k^\star = 20$
$t = 1$	$0, 20$	$20(1+r), 0$
$t = t^\star = 0.5$	$0, 20$	$10(1+r), 10(1+r)$

Pure NE are $\{t = 1, k = 0\}$,$\{t = t^\star, k = 0\}$

Since $k = 0$ weakly dominates $k = k^\star$, and in this case t is irrelevant, every profile $(0, m)$ is a NE, where $m \in [0, 1]$ is shorthand for the mix with weight m on $t = 1$ and weight $(1-m)$ on $t = t^\star = 0.5$.

g) By definition, $d \equiv \frac{q}{1+i}$, where i is annual interest rate (assuming one harvest per year) while q reflects the chances of the govt remaining in power and the investor remaining solvent. Both parameters affect the players' patience.

h) For Government, if choose $t^\star = 0.5$, it will earn $10(1 + r)$ in each period; if choose $t^\star = 1$ it will earn $20(1 + r)$ at first stage and 0 afterwards.
$PV_d(10(1+r), 10(1+r), ...) = \sum_{t=0}^{\infty} d^t 10(1+r) = 10\frac{1+r}{1-d}$
$PV_d(20(1+r), 0, 0, ...) = 20(1+r)$
To sustain an efficient equilibrium in the repeated game, we need the first PV to be higher, so $10(1+r)\frac{1+r}{1-d} \geq 20(1+r)$, or $d \in [\frac{1}{2}, 1)$

i) From previous parts, we know that when $d \geq \frac{r}{r+1}$, the game is efficient, where $k^\star = 20$, $t^\star \leq \frac{r}{r+1}$.

[Note: Dan edited this answer.]

Problem #3

a) The NFG with p proportion of the population playing strategy U is showed below with the :

	U	D
U	$0, 0$	$4, 1$
D	$1, 4$	$2, 2$

The payoff difference is $D(p) = (1 - p)4 - [p + (1 - p)2] = 2 - 3p$. The solution to $D(p) = 0$ is $p^\star = \frac{2}{3}$, and $D'(p^*) < 0$, ie, it is a downcrossing.

Thus, evolutionary equilibrium is where the population share of U is $p^\star = \frac{2}{3}$.

b) Suppose the proportion of population 1 playing U is p, the proportion of population 2 playing U is q. Based on part a), when $D_2(p) < \frac{2}{3}$, U is better strategy for population 2, and D is better in the opposite case Thus q decreases iff $D_2(p) > \frac{2}{3}$, .

Similarly when $D_1(q) < \frac{2}{3}$, U is better strategy for population 1, so p increases; when $D_1(q) > \frac{2}{3}$, p decreases. This is shown in the phase portrait:

c) For one population situation, the difference is that q=p everywhere, so all the possible combination of p and q are on the 45 degree line. The phase portrait is below:

d) Based on previous parts, we can see that under circumstance where there is only one population, the situation is simple. The mixed NE with $p = \frac{2}{3}$ will be the selected equilibrium.
When there are two populations, we can see that the two pure NEs, {U,D} and {D,U}, are both stable. They are selected equilibrium. Pure NE emerges depends on the initial state of the system. If it is below the saddle path, then (1,0) will emerge, and (0,1) will emerge if initially above the saddle path.

Problem #4

a) We have two countries A and B trading. They will split the total gains of 10.

$g_1 + g_2 = 10$

$g_2 = 10 - g_1$

NBS quantity to maximize= $F = g_1(10 - g_1)$

We will find FOC with respect to g_1;

$$\frac{\partial F}{\partial g_1} = 10 - 2g_1 = 0 \implies g_1 = 5$$

So, $10 = g_1 + g_2 \implies g_1 = g_2 = 5$ is Nash Bargaining Solution.

b) A third country C will be added to trade. Characteristic function is as follows

$v(A) = v(B) = v(C) = 0$

$v(A, B) = 10$

$v(A, C) = 15$

$v(B, C) = 5$

$v(A, B, C) = 15$

c) A game is convex when the characteristic function v is supermodular;
$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T), \forall S \subseteq T \subseteq N\backslash\{i\}, \forall i \in N$
Now, we will look at the case where $v(S \cup \{i\}) = v(A \cup \{C\})$ and $v(T \cup \{i\}) - v(T) = v(A \cup B \cup C) - v(A \cup B)$
So,
$v(S \cup \{i\}) = v(A \cup \{C\}) = 15 - 0 = 15$
$v(S \cup \{i\}) = v(A \cup \{C\}) = 15 - 10 = 5$
In this case;
$v(S \cup \{i\}) - v(S) > v(T \cup \{i\}) - v(T)$.
Hence, game is not convex.

d) We will find the core. Draw the simplex $15 = g_A + g_B + g_C$, and eliminate the blocked imputations $10 > g_A + g_B$, or $0 < g_C \geq 5$.

Also, eliminate from the simplex imputations with $g_B + g_C < 5$ or $g_A > 10$, and finally the blocked imputations $g_A + g_C < vAC = 15$, or $g_B > 0$.

All that is left is $g_B = 0, g_C = 5, g_A = 10$, so core is the singleton (10,0,5).

d. To find the Shapley value, fill out the table as follows.

Trade Coalition	MC_A	MC_B	MC_C
ABC	0	10	5
ACB	0	0	15
BAC	10	0	5
BCA	10	0	5
CAB	15	0	0
CBA	10	5	0
Sum	45	15	30
Sharpley value	7.5	2.5	5

Problem #5

In this question, we are given a steel seller and a steel buyer, where $MC_s = 10$ and $MB_b = 210 - q$ for q^{th} units purchased. Price=p.

a) We will find payoff functions for buyer and seller.
For seller, we know that marginal cost of seller is 10. So,the payoff function will be:
$\Pi_s(p, q) = (p - 10)q$

For buyer, we have the marginal benefit function, we need to take integral of this function :
$\int_0^q (210 - q)dq = 210q - q^2/2$

We need to subtract pq from the total benefit function to get payoff function

$\Pi_b(p, q) = 210q - q^2/2 - pq$

b) When we look at first order conditions for seller and buyer payoff functions with respect to q and p. For buyer,

$$\frac{\partial \Pi_b(p, q)}{\partial q} = 210 - p - q \implies p = 210 - q$$

$$\frac{\partial \Pi_b(p, q)}{\partial p} = -q \leq 0$$

For seller,

$$\frac{\partial \Pi_s(p, q)}{\partial q} = p - 10 \geq 0$$

$$\frac{\partial \Pi_s(p, q)}{\partial p} = q \geq 0$$

So, we understand from buyer's FOC's that buyer will not play if $q \leq 0$, so price, given $p = 210 - q$ should be $p < 210$ and quantity should be $q \geq 0$.

From seller's FOC's, we understand that seller will not accept $p < 10$ and quantity should also be $q \geq 0$.

Given these rules, let's find the optimum point for buyer, given the fact that seller will not accept a price less than 10. So, buyer will offer $p = 10$:

Given this: $p = 10$
$10 = 210 - q$
$q = 200$
So, we get one set as (200,10).
Let's look at seller's optimum given buyer's FOC's:

So, we will take quantity as given in buyer's payoff function $= q = 210 - p$

Seller will maximize:

$$\Pi_s = (210 - p)(p - 10)$$

Taking derivative with respect to quantity

$$\frac{\partial \Pi_s(p, q)}{\partial p} = 220 - 2p$$

$$\implies p = 110$$

So, seller will set $p = 110$, given this price quantity $q = 210 - p \implies q = 100$

The other set is (100,110)

Let's draw the pareto frontier given these sets;

Now, let's look at payoffs:

Plug in $p = 210 - q$ back to profit equations:

$\Pi_s(q) = q(200 - q) = -q^2 + 200q$

$\Pi_b(q) = 1/2q^2$

Then we can get:

$\Pi_s = -2\Pi_b + 200\sqrt{2\Pi_b}$

We found that buyer's optimum point is (200,10)

$\Pi_b(q) = 210(200) - (200)^2/2 - (10)(200)$

Given $\Pi_b(p, q) = 210(200) - (200)^2/2 - (10)(200)$

$\Pi_b(p, q) = 20000$

At this point, seller's payoff is;

$\Pi_s(p, q) = (10 - 10)200 = 0$

Seller's optimum point is (100,110)

$\Pi_s(p, q) = (110 - 10)100$

$\Pi_s(p, q) = 1000$

Buyer's payoff at this set is

$\Pi_b(p, q) = 210(100) - (100)^2/2 - (110)(100) = 5000$

So, buyer will not accept any payoff below 5000 and seller will not accept any payoff below 0.

c)Actually, we solved for this part in part b, let's remember what we did;

So, seller is going to set the price. Given that buyer will choose quantity $q = 210 - p$, using this equation in their payoff function, seller will maximize:

$\Pi_s(p,q) = (p-10)(210-p)$

$$\frac{\partial \Pi_s(p,q)}{\partial p} = 220 - 2p$$

We find $p = 110$ and $q = 100$

$\Pi_s(p,q) = (p-10)(210-p) = 10000$

$\Pi_b(p,q) = 210q - q^2/2 - pq = 5000$

d) Buyer will choose the lowest price they can choose, which is 10.

$\Pi_b(p,q) = 210q - q^2/2 - pq$

$$\frac{\partial \Pi_b(p,q)}{\partial q} = 210 - p - q \implies p = 210 - q$$

So, $p = 10$ and $q = 200$

$\Pi_b(p,q) = 210q - q^2/2 - pq = 20000$

$\Pi_s(p,q) = (p-10)(210-p) = 0$

e) We will find Nash Bargaining;

First, if utility is transferable:

Buyer will not play if $\Pi_b < 5000$, so minimum payoff of buyer is 5000.

Maximum payoff of buyer is when $p = 10$ $q = 200$, $\Pi_b = 20000$

Given these,maximization problem is

$maxF = (\Pi_b - 5000)(20000 - \Pi_b)$

Taking FOC with respect to Π_b

$25000 - 2\Pi_b^* = 0$

$\Pi_b^* = 12500$

Seller payoff is $= 20000 - 12500 = 7500$

Second, if utility is not transferable:

$maxF = (\Pi_b - 5000)(\Pi_s - 0)$

It can be written as:

$maxF = (\Pi_b - 5000)(-2\Pi_b + 200\sqrt{2\Pi_b})$

Taking FOC with respect to Π_b

$-4\Pi_b + 300\sqrt{2\Pi_b} - 500000\sqrt{2/\Pi_b} + 10000 = 0$

$\Pi_b^* \simeq 13090$

Seller payoff is $\simeq -2 * 13090 + 200\sqrt{2 * 13090} = 6180$