# PS2 Advanced Microeconomics: Answer Key

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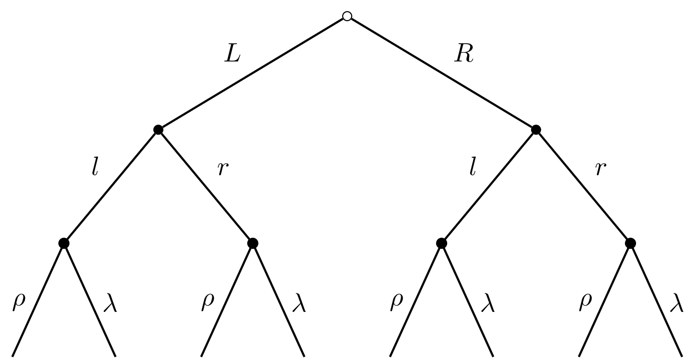
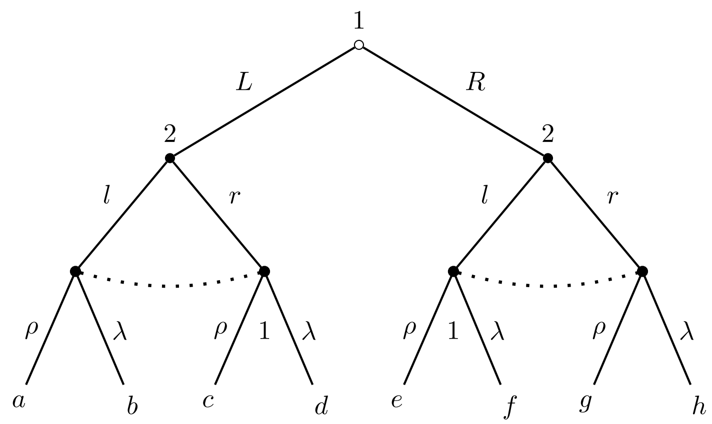
## Question 1

### Part a




### Part b



**Part c**

- Complete set of strategies for:
  - **Player 1**: $(R\rho\lambda), (R\rho\rho), (R\lambda\lambda), (R\lambda\rho), (L\rho\lambda), (L\rho\rho), (L\lambda\lambda), (L\lambda\rho)$
  - **Player 2**: $(rr), (rl), (lr), (ll)$
- Full normal form:

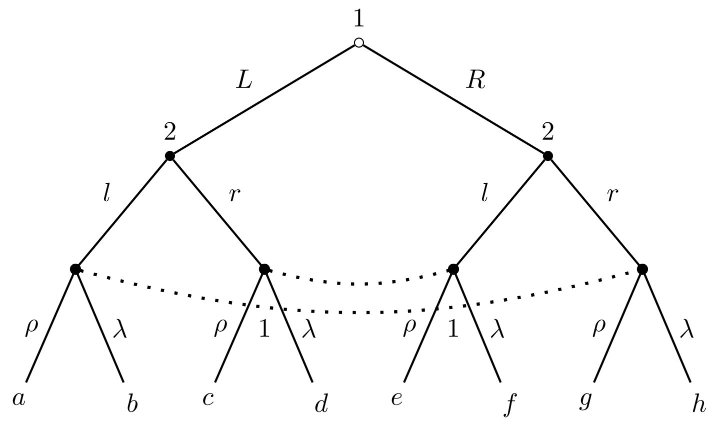
| | $rr$ | $rl$ | $ll$ | $lr$ |
|---|---|---|---|---|
| $R\rho\rho$ | a | a | c | c |
| $R\rho\lambda$ | a | a | c | c |
| $R\lambda\rho$ | b | b | d | d |
| $R\lambda\lambda$ | b | b | d | d |
| $L\rho\rho$ | e | g | g | e |
| $L\rho\lambda$ | f | h | h | f |
| $L\lambda\rho$ | e | g | g | e |
| $L\lambda\lambda$ | f | h | h | f |

- Reduced normal form (collapsing identical rows, using $*$ as wildcard symbol):

| | $rr$ | $rl$ | $ll$ | $lr$ |
|---|---|---|---|---|
| $R\rho*$ | a | a | c | c |
| $R\lambda*$ | b | b | d | d |
| $L*\rho$ | e | g | g | e |
| $L*\lambda$ | f | h | h | f |

  No identical columns, so no further reduction.

### Part d




Here, player 2 is randomizing between $r$ & $l$ with probabilities $(\sigma_2)$ & $(1-\sigma_2)$ respectively while Player 1 is randomizing between matched (i.e $(Rr, Ll)$) and unmatched (i.e. $(Rl, Lr)$) with probabilities $(\sigma_1)$ & $(1-\sigma_1)$ respectively.

$$P(\rho|Rr, Ll) = \frac{P(\rho, (Rr, Ll))}{P(Rr, Ll)}$$

$$= \frac{\sigma_1(R\rho)\sigma_2(r) + \sigma_1(L\rho)(1-\sigma_2)(l)}{[\sigma_1(R\rho)\sigma_2(r) + \sigma_1(L\rho)(1-\sigma_2)(l)] + [\sigma_1(R\lambda)\sigma_2(r) + \sigma_1(L\lambda)(1-\sigma_2)(l)]}$$

### Part e

(extra credit, skip if pressed for time.)
According to Kuhn's theorem: In a finite extensive form game with perfect recall:

1. Each behavioural strategy has an outcome-equivalent mixed strategy
2. Each mixed strategy has an outcome equivalent behavioural strategy

For the game in part (b), based on the mixed strategy profiles $\sigma_1$ & $\sigma_2$, player 1 can compute behavioural strategy for each information set.
If Player 2 randomizes with the probabilities q1 for (r,r), q2 for (r,l), q3 for (l,r) & q4 for (l,l) and Player 1 randomizes with probabilities p1 for $(R, \rho)$, p2 for $(R, \lambda)$, p3 for $(L, \rho)$ & p4 for $(L, \lambda)$:

$$\begin{aligned} P(\rho|(Info \quad set \quad 1:(r|R,l|R))) &= \frac{P(\rho,r,R)+P(\rho,l,R)}{P(R,r)+P(R,l)} \\ &= \frac{p1\cdot(q1+q2+q3+q4)}{(p1+p2)\cdot(q1+q2+q3+q4)} \\ &= \frac{p1}{p1+p2} \end{aligned}$$

$$\begin{aligned} P(\lambda|(Info \quad set \quad 1:(r|R,l|R))) &= \frac{P(\lambda,r,R)+P(\lambda,l,R)}{P(R,r)+P(R,l)} \\ &= \frac{p2\cdot(q1+q2+q3+q4)}{(p1+p2)\cdot(q1+q2+q3+q4)} \\ &= \frac{p2}{p1+p2} \end{aligned}$$

Similarly,

$$\begin{aligned} P(\rho|(Info \quad set \quad 2:(r|L,l|L))) &= \frac{P(\rho,r,L)+P(\rho,l,L)}{P(L,r)+P(L,l)} \\ &= \frac{p1\cdot(q1+q2+q3+q4)}{(p1+p2)\cdot(q1+q2+q3+q4)} \\ &= \frac{p3}{p3+p4} \end{aligned}$$

$$\begin{aligned} P(\lambda|(Info \quad set \quad 2:(r|L,l|L))) &= \frac{P(\lambda,r,L)+P(\lambda,l,L)}{P(L,r)+P(L,l)} \\ &= \frac{p2\cdot(q1+q2+q3+q4)}{(p1+p2)\cdot(q1+q2+q3+q4)} \\ &= \frac{p4}{p4+p2} \end{aligned}$$

Hence, we can induce Player 1's behaviour strategy at each info set from Player 1's mixed strategy.

In part (d), player 1's behaviour strategy when there is imperfect recall is given by:

$$P(\rho|matched) = \frac{p1 \cdot (q1 + q2) + p3 \cdot (q2 + q4)}{(p1 + p2) \cdot (q1 + q2) + (p3 + p4) \cdot (q2 + q4)}$$

$$P(\lambda|matched) = \frac{p2 \cdot (q1 + q2) + p4 \cdot (q2 + q4)}{(p1 + p2) \cdot (q1 + q2) + (p3 + p4) \cdot (q2 + q4)}$$

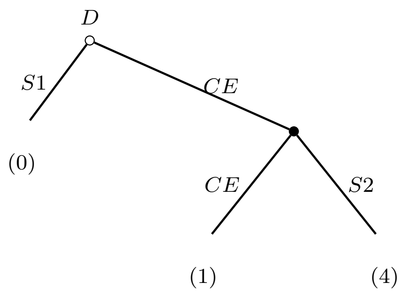
$$P(\rho|unmatched) = \frac{p1 \cdot (q3 + q4) + p3 \cdot (q1 + q3)}{(p1 + p2) \cdot (q2 + q4) + (p3 + p4) \cdot (q1 + q3)}$$

$$P(\lambda|unmatched) = \frac{p2 \cdot (q3 + q4) + p4 \cdot (q1 + q3)}{(p1 + p2) \cdot (q3 + q4) + (p3 + p4) \cdot (q1 + q3)}$$

Therefore, the induced behavioural strategy for games (b) and (d) are different. For game (d), player 1 can no longer induce behavioural strategy from his own mixed strategy alone. Information on player 2's mixed strategy is also required to induce behavioural strategy. When Player 2's mixed strategy changes, player 1's behavioural strategy also needs to change and hence, we cannot induce a unique behavioural strategy in part (d).

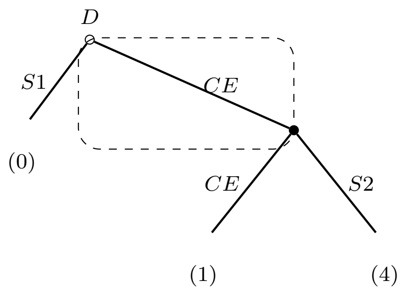
## Question 2

### Part a



The optimal strategy for the driver is to turn south at junction 2 ($S2$).

**Part b**




The information set is drawn as above because the driver does not know if she turned or continued east at the first junction.

**Part c**

p = probability that the driver turned (on any junction visible) To optimize, note that expected payoff then is

$$\begin{aligned} EP = p \cdot (0) + 1 \cdot (1-p)^2 + 4 \cdot (p) \cdot (1-p) \\ = (1-p)^2 + 4 \cdot (p - p^2) \\ \frac{\partial EP}{\partial P} = -2 \cdot (1-p) + 4 \cdot (1-2p) \quad = \quad 0 \\ \Rightarrow p^* = \frac{1}{3} \\ \Rightarrow EP^* = \frac{4}{3} \end{aligned}$$

For comparison, the optimal pure strategy in this case would be the following:

- Upon facing a junction, since the driver does not know whether it is the first or second, she decides to avoid the possibility of payoff equal to 0
- Hence, she chooses to continue east without driving through any junction, obtaining payoff $1 < \frac{4}{3}$.

### Part d

Suppose the driver is committed to the optimal mixed strategy $p^* = \frac{1}{3}$. The likelihood she reaches the first node is 1 and the likelihood of reaching the second node is $1 - p^*$, so the Bayesian posterior probability that she is at the first node is $1/(1 + 1 - p^*) = 1/(2 - p^*) = 0.6$. If she deviates from the plan and turns with probability 1, she increases expected payoff to $0(.6) + 4(.4) > 1.6 > \frac{4}{3}$. This incentive to deviate from the optimal plan is a version of time inconsistency. Of course, if she actually might deviate, the probability that she is at the first not 0.6 after all, so the paradox continues...

## Question 3

### (a)

| | h | t |
|---|---|---|
| H | $\underline{a}$,0 | 0,$\underline{d}$ |
| T | 0,$\underline{c}$ | $\underline{b}$,0 |

- Since $a, b, c, d > 0$, Row's and Column's best responses are

$$B_R(h) = H, B_R(t) = T,$$

$$B_C(H) = t, B_C(T) = h$$

- We find no strategy profile where two players' best responses overlap. Hence, there is no pure strategy NE.
- Next we search for mixed strategy Nash equilibria.
  - Let Row's mixed strategy be $\sigma_R = \{p, 1-p\} \in \Delta$. (p is probability of playing H; 1-p is probability of playing T).
  - Let Column's mixed strategy be $\sigma_C = \{q, 1-q\} \in \Delta$. (q is probability of playing h; 1-q is probability of playing t).
- Row's expected payoffs are
  - $E[H|\sigma_C] = aq + 0(1-q) = aq$
  - $E[T|\sigma_C] = 0q + b(1-q) = b - bq$

$$E[H|\sigma_C] \geq E[T|\sigma_C]$$

$$\Leftrightarrow aq \geq b - bq$$
$$\Leftrightarrow (a+b)q \geq b$$
$$\Leftrightarrow q \geq \frac{b}{a+b}$$

Row's best response is

$$B_R(\sigma_C) = \begin{cases} \{H\} = \{p = 1\} \quad (q > \frac{b}{a+b}) \\ \{T\} = \{p = 0\} \quad (q < \frac{b}{a+b}) \\ \{H, T, p \in [0,1]\} \quad (q = \frac{b}{a+b}) \end{cases}$$

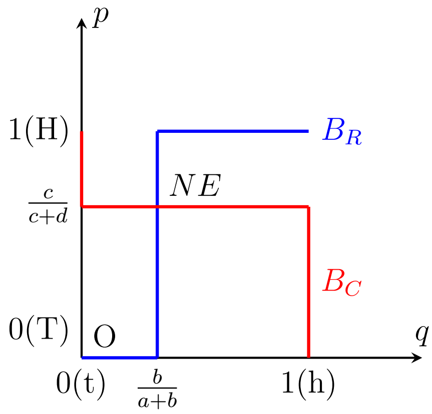
- Column's expected payoffs are
  - $E[h|\sigma_R] = 0p + c(1-p) = c - cp$
  - $E[t|\sigma_R] = dp + 0(1-q) = dp$

$$E[h|\sigma_R] \geq E[t|\sigma_R]$$
$$\Leftrightarrow c - cp \geq dp$$
$$\Leftrightarrow c \geq (c+d)p$$
$$\Leftrightarrow \frac{c}{c+d} \geq p$$

Column's best response is

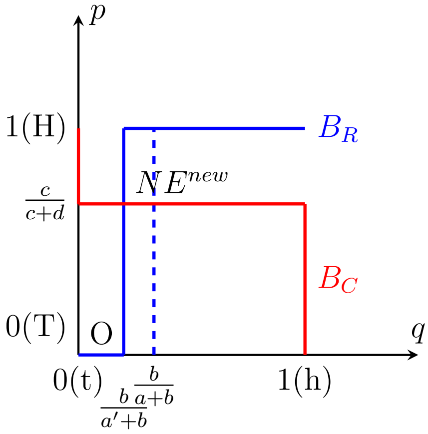
$$B_C(\sigma_R) = \begin{cases} \{h\} = \{q = 1\} \quad (p < \frac{c}{c+d}) \\ \{t\} = \{q = 0\} \quad (p > \frac{c}{c+d}) \\ \{h, t, q \in [0,1]\} \quad (p = \frac{c}{c+d}) \end{cases}$$

- Mapping Row's and Column's best response, we find that $(\sigma_R{}^*, \sigma_C{}^*) = ((p^* = \frac{c}{c+d}, 1 - p^* = \frac{d}{c+d}), (q^* = \frac{b}{a+b}, 1 - q^* = \frac{a}{a+b}))$ is a mixed strategy NE.

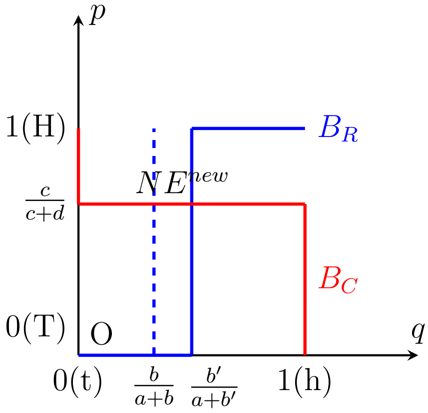


**(b) Comparative Statics**

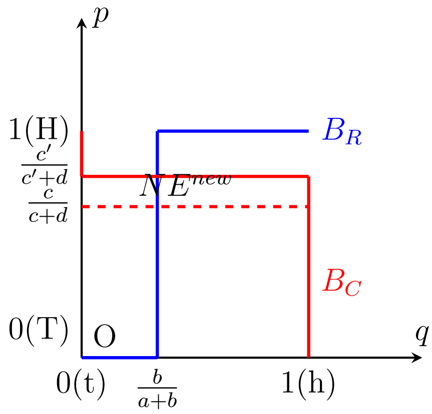
- Case1: a increases to a'
  - Column's optimal q decreases to $\frac{b}{a'+b}$
  - New mixed NE is $(\sigma_R{}^*, \sigma_C{}^*) = (\frac{c}{c+d}, \frac{d}{c+d}), (\frac{b}{a'+b}, \frac{a'}{a'+b}))$



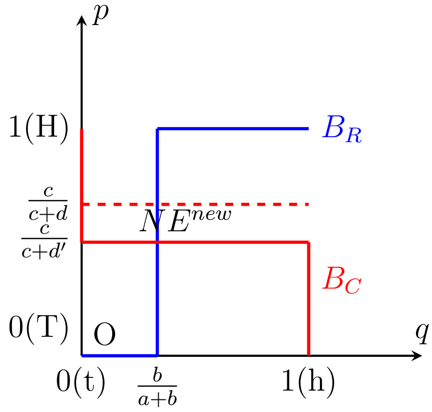

- Case2: b increases to b'
  - Column's optimal q increases to $\frac{b'}{a+b'}$
  - New mixed NE is $(\sigma_R{}^*, \sigma_C{}^*) = (\frac{c}{c+d}, \frac{d}{c+d}), (\frac{b'}{a+b'}, \frac{a}{a+b'}))$




- Case3: c increases to c'
  - Row's optimal p increases to $\frac{c'}{c'+d}$
  - New mixed NE is $(\sigma_R{}^*, \sigma_C{}^*) = (\frac{c'}{c'+d}, \frac{d}{c'+d}), (\frac{b}{a+b}, \frac{a}{a+b}))$




- Case4: d increases to d'
  - Row's optimal p decreases to $\frac{c}{c+d'}$
  - New mixed NE is $({\sigma_R}^*, {\sigma_C}^*) = ((\frac{c}{c+d'}, \frac{d'}{c+d'}), (\frac{b}{a+b}, \frac{a}{a+b}))$




### (c) Interpretation of Comparative Statics

- At the mixed strategy NE, each player chooses a mixed strategy ${\sigma_i}^*$ such that ${\sigma_i}^*$ makes its opponents being indifferent among the available strategies. (Otherwise, the opponents have an incentive to choose a pure strategy that gives the highest expected payoff with probability 1.) Therefore, changes in player $-i$'s payoff affect player $i$'s mixed strategy.

## Question 4

### (a)

| | L | C | R |
|---|---|---|---|
| T | 2,0 | 1,1 | 4,2 |
| M | 3,4 | 1,2 | 2,3 |
| B | 1,3 | 0,2 | 3,0 |

$\Longrightarrow$

| | L | C | R |
|---|---|---|---|
| T | 2,0 | 1,1 | 4,2 |
| M | 3,4 | 1,2 | 2,3 |

- Row's strategy B is strictly dominated by T. We remove row B.
- After removing row B, Column's strategy C is strictly dominated by R. We remove C.

| | L | R |
|---|---|---|
| T | 2,0 | <u>4,2</u> |
| M | <u>3,4</u> | 2,3 |

- After the iterated deletion of dominated strategies, we have a two by two matrix.
- Searching for strategy profiles that are mutual best response for each player, we find that (M, L) and (T, R) are two pure strategy Nash equilibria.
- Next we search for mixed strategy Nash equilibrium.
  - Let Row's mixed strategy be $\sigma_R = \{p, 1-p\} \in \Delta$. (p is probability of playing T; 1-p is probability of playing M).
  - Let Column's mixed strategy be $\sigma_C = \{q, 1-q\} \in \Delta$. (q is probability of playing L; 1-q is probability of playing R).
- Row's expected payoffs are
  - $E[T|\sigma_C] = 2q + 4(1-q) = -2q + 4$
  - $E[M|\sigma_C] = 3q + 2(1-q) = q + 2$

  $$E[T|\sigma_C] \geq E[M|\sigma_C] \Leftrightarrow -2q + 4 \geq q + 2 \Leftrightarrow 2 \geq 3q \Leftrightarrow 2/3 \geq q$$

  Row's best response is

  $$B_R(\sigma_C) = \begin{cases} \{T\} = \{p = 1\} \;\; (q < \frac{2}{3}) \\ \{M\} = \{p = 0\} \;\; (q > \frac{2}{3}) \\ \{T, M, p \in [0,1]\} \;\; (q = \frac{2}{3}) \end{cases}$$

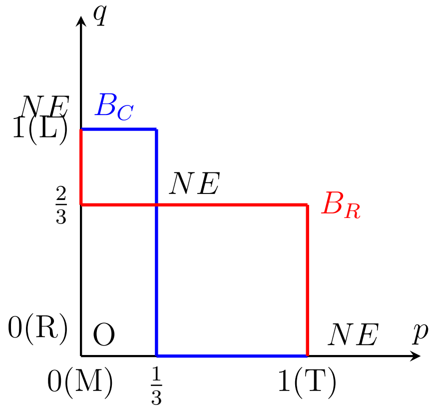
- Column's expected payoffs are
  - $E[L|\sigma_R] = 0p + 4(1-p) = -4p + 4$
  - $E[R|\sigma_R] = 2p + 3(1-p) = -p + 3$

$$E[L|\sigma_R] \geq E[R|\sigma_R] \Leftrightarrow -4p + 4 \geq -p + 3 \Leftrightarrow 1 \geq 3p \Leftrightarrow 1/3 \geq p$$

Column's best response is

$$B_C(\sigma_R) = \begin{cases} \{L\} = \{q = 1\} & (p < \frac{1}{3}) \\ \{R\} = \{q = 0\} & (p > \frac{1}{3}) \\ \{L, R, q \in [0,1]\} & (p = \frac{1}{3}) \end{cases}$$

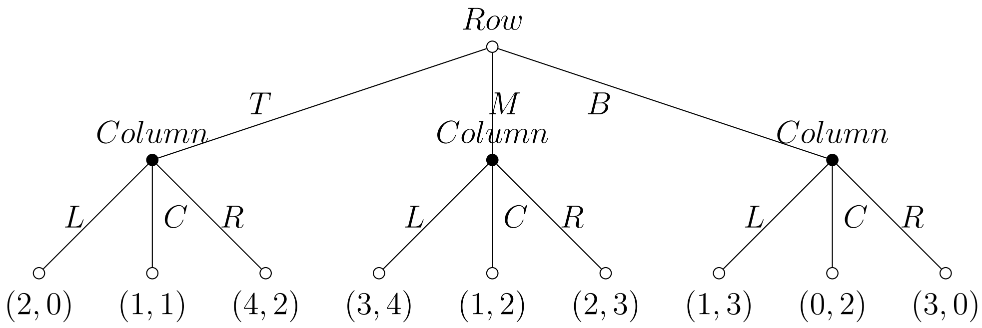
- Mapping Row's and Column's best response, we find that $(\sigma_R{}^*, \sigma_C{}^*) = ((p^* = \frac{1}{3}, 1 - p^* = \frac{2}{3}), (q^* = \frac{2}{3}, 1 - q^* = \frac{1}{3}))$ is a mixed strategy NE.
  - Column chooses q such that Row is indifferent between T and B.
  - Row chooses p such that Column is indifferent between L and R.




- Note that we can also find the two pure strategy NE ((M,L) and (T,R)) in the graph.

**(b)**

- Assuming Row moves first and Column moves second under perfect information, we can construct the following extensive form game:




- In order to find NE, we translate this tree into a normal form representation (See next page).

| | | Column | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | LLL | LLC | LLR | LCL | LCC | LCR | LRL | LRC | LRR | CLL | CLC | CLR | CCL | CCC | CCR | CRL | CRC | CRR | RLL | RLC | RLR | RCL | RCC | RCR | RRL | RRC | RRR | |
| Row | T | 2,0 | 2,0 | 2,0 | 2,0 | 2,0 | 2,0 | 2,0 | 2,0 | 2,0 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | |
| | M | 3,4 | 3,4 | 3,4 | 1,2 | 1,2 | 1,2 | 2,3 | 2,3 | 2,3 | 3,4 | 3,4 | 3,4 | 1,2 | 1,2 | 1,2 | 2,3 | 2,3 | 2,3 | 3,4 | 3,4 | 3,4 | 1,2 | 1,2 | 1,2 | 2,3 | 2,3 | 2,3 | |
| | B | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 | |
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | |
| | | | | | | 1 | 1 | | 1 | 1 | | | | | 19 | 19 | | 19 | 19 | | | | | | | | | | Dominated by |

*Row player does not have any strictly dominated strategy.

| | | Column | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | LLL | LLC | LLR | LCL | LRL | CLL | CLC | CLR | CCL | CRL | RLL | RLC | RLR | RCL | RCC | RCR | RRL | RRC | RRR |
| Row | T | 2,0 | 2,0 | 2,0 | 2,0 | 2,0 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 |
| | M | 3,4 | 3,4 | 3,4 | 1,2 | 2,3 | 3,4 | 3,4 | 3,4 | 1,2 | 2,3 | 3,4 | 3,4 | 3,4 | 1,2 | 1,2 | 1,2 | 2,3 | 2,3 | 2,3 |
| | B | 1,3 | 0,2 | 3,0 | 1,3 | 1,3 | 1,3 | 0,2 | 3,0 | 1,3 | 1,3 | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 |
| | | 1 | 2 | 3 | 4 | 7 | 10 | 11 | 12 | 13 | 16 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |

Eliminated strictly dominated columns.
There is no more strictly dominated strategy.

| | | Column | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | LLL | LLC | LLR | LCL | LRL | CLL | CLC | CLR | CCL | CRL | RLL | RLC | RLR | RCL | RCC | RCR | RRL | RRC | RRR |
| Row | T | 2,0 | 2,0 | 2,0 | 2,0 | 2,0 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 |
| | M | 3,4 | 3,4 | 3,4 | 1,2 | 2,3 | 3,4 | 3,4 | 3,4 | 1,2 | 2,3 | 3,4 | 3,4 | 3,4 | 1,2 | 1,2 | 1,2 | 2,3 | 2,3 | 2,3 |
| | B | 1,3 | 0,2 | 3,0 | 1,3 | 1,3 | 1,3 | 0,2 | 3,0 | 1,3 | 1,3 | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 |
| | | 1 | 2 | 3 | 4 | 7 | 10 | 11 | 12 | 13 | 16 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |

Yellow is Row's best response.

*Italic is Column's best response.*

| | | Column | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | LLL | LLC | LLR | LCL | LRL | CLL | CLC | CLR | CCL | CRL | RLL | RLC | RLR | RCL | RCC | RCR | RRL | RRC | RRR |
| Row | T | 2,0 | 2,0 | 2,0 | 2,0 | 2,0 | 1,1 | 1,1 | 1,1 | 1,1 | 1,1 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 | 4,2 |
| | M | 3,4 | 3,4 | 3,4 | 1,2 | 2,3 | 3,4 | 3,4 | 3,4 | 1,2 | 2,3 | 3,4 | 3,4 | 3,4 | 1,2 | 1,2 | 1,2 | 2,3 | 2,3 | 2,3 |
| | B | 1,3 | 0,2 | 3,0 | 1,3 | 1,3 | 1,3 | 0,2 | 3,0 | 1,3 | 1,3 | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 | 1,3 | 0,2 | 3,0 |
| | | 1 | 2 | 3 | 4 | 7 | 10 | 11 | 12 | 13 | 16 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |

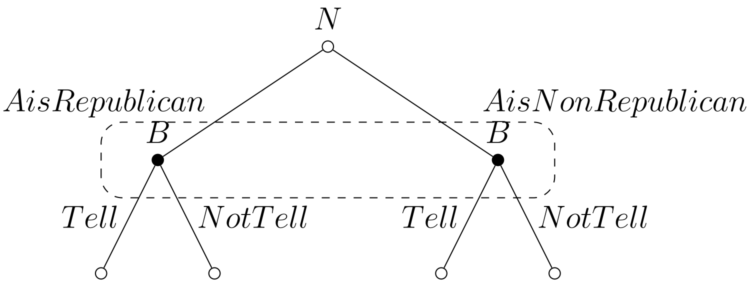
There are 16 Nash equilibria.

Green are NE (where best response of Row and Column are overlapped).

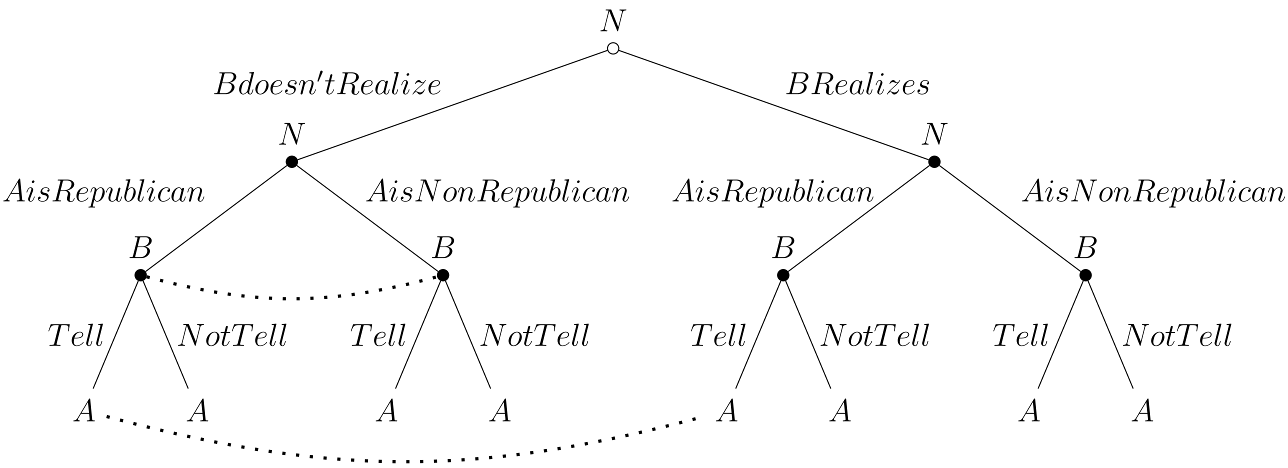
Figure 1: Normal form game representation

## Question 5

**(a)**




**(b)**




Note: This is the Harsanyi conversion trick applied to higher order beliefs. The game on the left represents the type "B doesn't realize," the game on the right represents the type "B realizes," and the initial Nature move and A's info set at the bottom tie these games together. The interpretation of the tree is that A does not know, after hearing the joke, which type she is dealing with, although she does know that she is indeed a Republican, etc.

## 8.D.5

### (a)

Suppose the two locations are $x_1$ and $x_2$.   $x_1$ ,$x_2 \in [0, 1]$. We need consider the two vendors' payoff functions under all the conditions

(i) $x_1 < x_2$
All of the customers on the left side of $\frac{x_1+x_2}{2}$ will buy ice-cream from vendor 1. All of the customers on the right side of $\frac{x_1+x_2}{2}$ will buy ice-cream from vendor 2. Then the payoff functions of vendor 1 and 2 are:

$$f_1(x_1, x_2) = \frac{x_1 + x_2}{2}$$

$$f_2(x_1, x_2) = 1 - \frac{x_1 + x_2}{2}$$

(ii) $x_2 < x_1$
All of the customers on the left side of $\frac{x_1+x_2}{2}$ will buy ice-cream from vendor 2. All of the customers on the right side of $\frac{x_1+x_2}{2}$ will buy ice-cream from vendor 1. Then the payoff functions of vendor 1 and 2 are:

$$f_1(x_1, x_2) = 1 - \frac{x_1 + x_2}{2}$$
$$f_2(x_1, x_2) = \frac{x_1 + x_2}{2}$$

(iii)$x_1 = x_2$

The two vendors split the customers evenly.

$$f_1(x_1, x_2) = f_2(x_1, x_2) = \frac{1}{2}$$

Therefore, the payoff functions of the two vendors are:

$$f_1(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2} & x_1 < x_2 \\ \frac{1}{2} & x_1 = x_2 \\ 1 - \frac{x_1+x_2}{2} & x_1 > x_2 \end{cases}$$

$$f_2(x_1,x_2) = \begin{cases} 1-\frac{x_1+x_2}{2} & x_1 < x_2 \\ \frac{1}{2} & x_1 = x_2 \\ \frac{x_1+x_2}{2} & x_1 > x_2 \end{cases}$$

We can prove $f_1(x_1, x_2) = f_2(x_1, x_2) = \frac{1}{2}$ is the unique Nash Equilibrium by proving that all the other locations are not stable:
(i) If $x_1 = x_2 < \frac{1}{2}$
Both vendors have incentives to move to the right side to gain more customers.
(ii) If $x_1 = x_2 > \frac{1}{2}$
Both vendors have incentives to move to the left side to gain more customers.
(iii) If $x_1 < x_2$
Both vendors have incentives to move to each other to gain more customers.
(iv) If $x_1 > x_2$
Both vendors have incentives to move to each other to gain more customers.

Therefore, $x_1 = x_2 = \frac{1}{2}$ is the unique NE.

## (b)

Suppose there exists a pure NE $(x_1^*, x_2^*, x_3^*)$,
(i) If $x_1^* = x_2^* = x_3^*$
Each vendor have incentives to move right or left to gain more customers, thus it's not NE.
(ii) If $x_1^* = x_2^* < x_3^*$
Vendor 3 can move to left side to gain more customers, thus it's not NE.
(iii) If $x_1^* = x_2^* > x_3^*$
Vendor 3 can move to right side to gain more customers, thus it's not NE.
(iv) If $x_1^* \neq x_2^* \neq x_3^*$
The vendors on the left and right both can gain more customers by moving towards the middle, thus it's not NE.

Therefore, there is no pure startegy NE under this situation. (It would take some explanation to interpret a mixed NE. It turns out that mixed NE does exist in this case, even though the payoff function is discontinuous.)

## 8.D.9

| | LL | L | M | R |
|---|---|---|---|---|
| U | 100,2 | -100,1 | 0,0 | -100,-100 |
| D | -100,-100 | 100,-49 | 1,0 | 100,2 |

### (a)

Since the payoff information is common knowledge, and I believe player 1 is as rational as I am, it seems I should choose LL or R. Both LL and R are part of Pareto efficient NE. (Not Pareto dominant NE; there is at most one of those.)

However, since I am not sure whether player 1 will choose U or D (since there are two pure NE), and payoff is so negative that it's too risky to choose LL or R, I will prefer M. M is safe strategy for me.

The uncertainty of player's action is formally related to "Trembling Hand" from Selten.

### (b)

From the NFG, we can find two pure NEs: {U,LL}, {D,R}.

For the mixed NE:

- Suppose player 2 chooses LL with probability $p_1$, chooses L with probability $p_2$, chooses M with probability $p_3$, chooses R with probability $p_4$. We know $p_4$=1-$p_1$-$p_2$-$p_3$.

  The expected payoff of player 1:

  - $Ef_1(U) = 100p_1 - 100p_2 - 100(1 - p_1 - p_2 - p_3)$
  - $Ef_1(D) = -100p_1 + 100p_2 + p_3 - 100(1 - p_1 - p_2 - p_3)$

Player 1's BR is:

$$B_1(\sigma_2) = \begin{cases} \{U\} \ \ if \ \ p_1 + \frac{199}{400}p_3 > \frac{1}{2} \\ \{D\} \ \ if \ \ p_1 + \frac{199}{400}p_3 < \frac{1}{2} \\ \{U, D, q \in [0,1]\} \ \ if \ \ p_1 + \frac{199}{400}p_3 = \frac{1}{2} \end{cases}$$

- Suppose player 1 chooses U with probability q,
  The expected payoff of player 2:

  - $Ef_2(LL) = 2q - 100(1-q)$
  - $Ef_2(L) = q - 49(1-q)$
  - $Ef_2(M) = 0$
  - $Ef_2(R) = -100q + 2(1-q)$

  Player 2's BR is:

$$B_2(\sigma_1) = \begin{cases} \{LL\} = \{p_1 = 1\} \ \ if \ \ \frac{51}{52} < q \leqslant 1 \\ \{LL, L, p \in \{(p_1,p_2)|p_1,p_2 \in [0,1], p_1+p_2 = 1\}\} \ \ if \quad q = \frac{51}{52} \\ \{L\} = \{p_2 = 1\} \ \ if \ \ \frac{49}{50} < q < \frac{51}{52} \\ \{LL, M, p \in \{(p_2,p_3)|p_2,p_3 \in [0,1], p_2+p_3 = 1\}\} \ \ if \quad q = \frac{49}{50} \\ \{M\} = \{p_3 = 1\} \ \ if \ \ \frac{1}{51} < q < \frac{49}{50} \\ \{M, R, p \in \{(p_3,p_4)|p_3,p_4 \in [0,1], p_3+p_4 = 1\}\} \ \ if \quad q = \frac{1}{51} \\ \{R\} = \{p_4 = 1\} \ \ if \ \ 0 \leqslant q < \frac{1}{51} \end{cases}$$

- Matching the two players BRs, we have three NEs in the game.
  - Two of them are pure NE:$\{U, LL\}, \{D, R\}$
  - The mixed NE should satisfy $p_1 + \frac{199}{400}p_3 = \frac{1}{2}$, matching it with the BR of player2, only $\{LL, L, p \in \{(p_1,p_2)|p_1,p_2 \in [0,1], p_1+p_2 = 1\}\} \ \ if \quad q = \frac{51}{52}$ satisfies the condition. Therefore, the mixed NE is: $(q^* = \frac{51}{52}, 1-q^* = \frac{1}{52}; p_1^* = \frac{1}{2}, p_2^* = \frac{1}{2}, p_3^* = 0, p_4^* = 0)$

## (c)

My strategy in (a) is not a component of NE strategy profile. It is a rationalizable strategy. Because under some situations, M is the BR for player 2. For example, player 1 takes the mixed strategy of ($\frac{1}{2}$ U,$\frac{1}{2}$ D), then M is the BR for player 2.

**(d)**

If preplay communication were possible, I would try to achieve (U,LL) or (D,R), which are payoff dominant NEs.