Problem Set 2

Econ 200

1. A firm has production function ln y = (1/3)ln x_1 + (1/2)ln x_2. Input 1 is unchangeable for the moment at the level $x_1 = 8$. Prices are \$8 and \$5 respectively for inputs 1 and 2.

(a) (6 points) What is the firm's variable cost? fixed cost (if any)? total cost? marginal cost?

Solution: In the short run, total costs are the sum of variable costs and fixed costs:

$$c(y) = c_v(y) + F$$

Therefore, we can find variable costs and fixed costs from total costs $c(y)$ which is simply the cost function $c(w_1, w_2, y)$ where $w_1 = 8$ and $w_2 = 5$.

In order to derive cost function, the first step is to derive conditional factor demand function. The cost minimization problem is,

$$\min_{x_2 \geq 0} 8(8) + 5x_2 \quad \text{subject to} \quad \frac{1}{3}\ln x_1 + \frac{1}{2}\ln 8 \geq \ln y$$

Since the firm has no incentive to use more x_2 than necessary to achieve the fixed output level, the solution of this optimization problem is implicitly defined as

$$\frac{1}{3}\ln 8 + \frac{1}{2}\ln x_2^* = \ln y$$

By solving this equation explicitly for x_2 yields $x_2^*(8, 5, y) = 2^{-2}y^2$. The second step is to substitute this factor demand function into cost. This gives us our cost function:

$$c(8, 5, y) = 5 \cdot 2^{-2}y^2 + 64$$

implying that variable costs are $c_v(y) = 5 \cdot 2^{-2}y^2$ and fixed costs are $F = 64$. Marginal costs are

$$\text{MC}(y) = \frac{dc}{dy} = 10 \cdot 2^{-2}y = \frac{5}{2}y$$

(b) (2 points) What is the firm's supply function?

Solution: From the profit maximization problem, optimal output to produce is implicitly defined by $p = MC(y^*)$. However, not the whole part of the marginal cost curve is not necessarily supply curve. The firm's supply curve is the upward-sloping portion of the marginal cost curve that lies above the average variable cost curve (You should be able to explain why). In this problem, the marginal cost curve is always upward-sloping (and thus marginal cost is always higher than average variable cost). Therefore, the firm's supply function is given by its marginal cost:

$$p = \frac{5}{2}y \quad \Rightarrow \quad y = \frac{2}{5}p$$

(You should verify that this is the same supply function that you get when you apply Hotelling's Lemma to the profit function.)

(c) (6 points) Now assume that both inputs can be adjusted freely. What are the firm's conditional input demands? What is its average cost? Marginal cost? Supply function?

Solution: To solve

$$\min_{x_1, x_2 \geq 0} w_1 x_1 + w_2 x_2 \quad \text{subject to} \quad \ln y = \frac{1}{3}\ln x_1 + \frac{1}{2}\ln x_2$$

we use the Lagrangian

$$\mathcal{L} = w_1 x_1 + w_2 x_2 - \lambda(\frac{1}{3}\ln x_1 + \frac{1}{2}\ln x_2 - \ln y)$$

which gives the first-order conditions

$$w_1 - \lambda\frac{1}{3x_1} = 0$$
$$w_2 - \lambda\frac{1}{2x_2} = 0$$

When we combine these two conditions to get rid of λ, we get $3w_1x_1 = 2w_2x_2$ (make sure that you can see this is equivalent with the tangency condition). We substitute this back into the constraint to get the conditional factor demands:

$$\ln y = \frac{1}{3}\ln x_1 + \frac{1}{2}\ln\left(\frac{3w_1}{2w_2}x_1\right)$$
$$\Rightarrow \quad y = x_1^{\frac{1}{3}}\left(\frac{3w_1}{2w_2}x_1\right)^{\frac{1}{2}} = \left(\frac{3w_1}{2w_2}\right)^{\frac{1}{2}} x_1^{\frac{5}{6}}$$
$$\Rightarrow \quad x_1^*(w_1, w_2, y) = \left(\frac{2w_2}{3w_1}\right)^{\frac{3}{5}} y^{\frac{6}{5}},$$
$$x_2^*(w_1, w_2, y) = \left(\frac{3w_1}{2w_2}\right)^{\frac{2}{5}} y^{\frac{6}{5}}$$

With these conditional factor demands, the cost function is

$$\begin{aligned} c(w_1, w_2, y) &= w_1 x_1^* + w_2 x_2^* \\ &= \left[\left(\frac{2}{3}\right)^{\frac{3}{5}} + \left(\frac{3}{2}\right)^{\frac{2}{5}}\right] w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{6}{5}} \\ &= \left(\frac{2}{3}\right)^{\frac{3}{5}}\left(1 + \frac{3}{2}\right) w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{6}{5}} \\ &= \left(\frac{2}{3}\right)^{\frac{3}{5}} \frac{5}{2} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{6}{5}} \end{aligned}$$

Long-run average cost is

$$\text{LAC}(y) = \frac{c}{y} = \left(\frac{2}{3}\right)^{\frac{3}{5}} \frac{5}{2} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{1}{5}}$$

and long-run marginal cost is

$$\begin{aligned} \text{LMC}(y) &= \frac{dc}{dy} = \frac{6}{5}\left(\frac{2}{3}\right)^{\frac{3}{5}} \frac{5}{2} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{1}{5}} \\ &= 2^{\frac{3}{5}} 3^{\frac{2}{5}} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{1}{5}} \end{aligned}$$

The firm's supply function is once again given by its marginal cost:

$$p = 2^{\frac{3}{5}} 3^{\frac{2}{5}} w_1^{\frac{2}{5}} w_2^{\frac{3}{5}} y^{\frac{1}{5}}$$
$$\Rightarrow \quad y^*(w_1, w_2, p) = \frac{p^5}{72 w_1^2 w_2^3}$$

(Alternatively, we could have found the supply function by maximizing profit. You should verify that the same supply function is found by applying Hotelling's Lemma to the profit function!)

2. Suppose that each firm in an industry has long run total cost function $c(y_i) = y_i^3 - 9y_i^2 + 36y_i$, and they face industry demand curve $y_T = 200 - 10p$.

a. What is each firms marginal cost function? Average cost function? Fixed cost? Supply curve? (4pts)

b. What is long run competitive equilibrium price?Output per firm? Number of firms? (3pts)

c. Compute the producer surplus (PS), consumer surplus (CS) and total surplus (TS) at long run competitive equilibrium. (4 pts)

Solution: Individual cost function is given as $c(y_i) = y_i^3 - 9y_i^2 + 36y_i$

a) Since cost function is given, it is immediate to derive marginal cost function, average cost function, and fixed cost. Since there is no term which does not depend on y_i, fixed cost is 0 and this means that average variable cost equals average cost. Therefore,

$$FC = 0, \tag{1}$$

$$MC \equiv c'(y_i) = 3y_i^2 - 18y_i + 36, \tag{2}$$

$$AC \equiv \frac{c(y_i)}{y_i} = y_i^2 - 9y_i + 36 = AVC. \tag{3}$$

Supply function is a portion of the MC curve, but firms produce positive amount only if the profit from producing is higher than not producing. Firms earn higher profit from producing when price is higher than the lowest value of AVC. Since we can rewrite AVC as

$$AVC = y_i^2 - 9y_i + 36 = \left(y_i - \frac{9}{2}\right)^2 + \frac{63}{4}, \tag{4}$$

the lowest value of AVC is $\frac{63}{4}$ when $y_i = \frac{9}{2}$. You can also derive this using $MC = AVC$. Therefore, the inverse supply function is,

$$p = S_i^{-1}(y_i) = 3y_i^2 - 18y_i + 36 \quad \text{for } y_i \geq \tfrac{9}{2}. \tag{5}$$

To obtain the supply function, solve that equation for y_i in terms of p in the relevant range. Using the quadratic formula and simplifying, you get

$$y_i = S_i(p) = \begin{cases} 3 + (\frac{p}{3} - 3)^{\frac{1}{2}} & \text{if } p \geq \frac{63}{4}, \\ 0 & \text{if } p < \frac{63}{4}. \end{cases} \tag{6}$$

b) In the long run, as long as firms can earn positive profit, potential entrants enter the market. Therefore, in the long run, price equals the lowest value of AC. In this simple example, since $FC = 0$, the lowest value of AC equals to that of AVC, which we derived already. Therefore, equilibrium price, p^*, is

$$p^* = \frac{63}{4}, \tag{7}$$

and given this equilibrium price, each firm determines the amount of supply by the optimality condition $P = MC$. However, since MC goes through the minimum of AC and p^* is the minimum value of AC, we already know that the optimal amount to supply is,

$$y_i^* = \frac{9}{2}. \tag{8}$$

You can also find this value by $MC = AC$.
Aggregate supply function, $S(p)$, is the sum of individual supply functions. As we derive, if $p < \frac{63}{4}$, no firms enter the market and if $p = \frac{63}{4}$, every firm is willing to supply $y_i^* = \frac{9}{2}$. Therefore, aggregate supply function is perfectly elastic at $p = \frac{63}{4}$,

$$S(p) = \begin{cases} \infty & \text{if } p \geq \frac{63}{4}, \\ 0 & \text{if } p < \frac{63}{4}. \end{cases} \tag{9}$$

Given aggregate supply function and aggregate demand function, we can derive competitive equilibrium. Since demand equals supply at $p = \frac{63}{4}$, the competitive equilibrium price, p^*, is

$$p^* = \frac{63}{4}. \tag{10}$$

By substituting equilibrium price into demand function yields equilibrium quantity,

$$q^* = 200 - 10\left(\frac{63}{4}\right) = \frac{85}{2}. \tag{11}$$

Since each firm produces $y_i^* = \frac{9}{2}$ and aggregate output is $q^* = \frac{85}{2}$, the number of firms are

$$N = \frac{\frac{85}{2}}{\frac{9}{2}} = \frac{85}{9} < 10. \tag{12}$$

So, N=9 in LRCE, because a 10th firm would earn a negative profit by entering this market and sell the remaining amount (Verify this from the graph of AVC and break-even price).

Consumer surplus is the area between demand curve and equilibrium price, producer surplus is the area between equilibrium price and supply curve. In approximation, by assuming $p^* = \frac{63}{4}$,

$$CS = \frac{1}{2}(20 - \frac{63}{4}) * \frac{85}{2} = \frac{1445}{16}, \tag{13}$$

$$PS = \frac{1}{2}(\frac{63}{4} - \frac{63}{4})\frac{85}{2} = 0, \tag{14}$$

$$TS = CS + PS = \frac{1445}{16}. \tag{15}$$

3. You estimated a 3-factor translog cost function for a client. Write down a possible numerical result. That is, including (made-up) coefficients, about half which might be zero).

a. Check that the coefficients that you made up satisfy the main required conditions (homogeneity, etc)

Solution: Cost function $c(\mathbf{w}, y)$ must satisfy

1. Non-decreasing in $(\mathbf{w}, y)$,
2. Homogeneous of degree 1 in $\mathbf{w}$,
3. Concave in $\mathbf{w}$ (negative semi-definite Hessian),
4. Continuous and differentiable (any combination of logarithmic functions will satisfy this unless you put additional terms inside the logarithmic functions).

For simplicity, I will demonstrate the case with 2-factor. The extension to 3-factor case is immediate though we have more terms. A translog cost function assuming zero coefficients on own quadratic terms is

$$\ln c(\mathbf{w}, y) = a_0 + a_1 \sum_{i=1}^{2} \ln w_i + \frac{1}{2} \sum_{i=1}^{2} \sum_{i=1}^{2} b_{ij} \ln w_i \ln w_j + \alpha_1 \ln y. \tag{16}$$

For condition 2, when you multiply all the input prices by λ, both sides of equation have additional $\ln\lambda$ from the original equation only when $\sum_{i=1}^{2} a_i = 1$, $b_{ij} = b_{ji}$, $\sum_{j=1}^{2} b_{ij} = 0$ (To see this, when we multiply all the input prices by λ,

$$
\begin{aligned}
&a_0 + a_1 \ln \lambda w_1 + a_2 \ln \lambda w_2 \\
&\quad + \frac{1}{2} b_{11} (\ln \lambda w_2)^2 + \frac{1}{2} b_{12} \ln \lambda w_1 \ln \lambda w_2 + \frac{1}{2} b_{21} \ln \lambda w_1 \ln \lambda w_2 + \frac{1}{2} b_{22} (\ln \lambda w_2)^2 + \alpha_1 \ln y \\
&= a_0 + a_1 \ln w_1 + a_2 \ln w_2 + \sum_{i=1}^{2} a_i \ln \lambda + \frac{1}{2} \sum_{i=1}^{2} \sum_{i=1}^{2} b_{ij} \ln w_i \ln w_j + \alpha_1 \ln y \\
&\quad + \frac{1}{2} \sum_{i=1}^{2} \sum_{i=1}^{2} b_{ij} (\ln \lambda)^2 + b_{11} \ln \lambda \ln w_1 + \frac{1}{2} (b_{12} + b_{21}) \ln \lambda \ln w_1 \\
&\quad + \frac{1}{2} (b_{12} + b_{21}) \ln \lambda \ln w_2 + b_{22} \ln \lambda \ln w_2 .) \qquad (17)
\end{aligned}
$$

Under these restrictions, condition 1 is satisfied when

$$\frac{\partial \ln c}{\partial \ln w_1} = a_1 + b_{11} \ln w_1 + (1 - b_{11}) \ln w_2 > 0, \qquad (18)$$

$$\frac{\partial \ln c}{\partial \ln w_2} = (1 - a_1) + b_{11} \ln w_2 + (1 - b_{11}) \ln w_1 > 0. \qquad (19)$$

This is always true at $w_1 = w_2$ when $a_i > 0, \forall i$. For condition 4, we can check the eigenvalues of the matrix $\mathbf{M}$, which is a transformed Hessian matrix of second derivatives using cost shares and each element is a function of coefficients from the regression and cost shares. See Baum and Linz (2009) for further details.

b. write down the implied factor demand equations.

Solution: Use Shepherd's Lemma

$$\frac{\partial c\,(w_1, w_2, w_3, y)}{\partial w_i} = x_i^*\,(w_1, w_2, w_3, y)$$

c. Suppose instead that you estimated a translog profit function. How would you modify your answers to parts a and b?

Solution: Properties part is analogous to cost function. For factor demand, Use Shepherd's Lemma for profit functions

$$\frac{\partial \pi \left(w_1, w_2, w_3, p\right)}{\partial w_i} = -x_i^* \left(w_1, w_2, w_3, p\right)$$

4. Varian questions 4a. Question 4.6

Solution: Two-plant problem.

$$c\left(y\right) = \min\left\{4\sqrt{y_1} + 2\sqrt{y_2}; y_1 + y_2 \geq y\right\}$$

Since the cost is concave, rather than convex, the optimal solution will always occur at a boundary. If you derive optimality conditions without checked the second derivative, you will maximize cost (instead of minimizing). The solution is to produce entirely at the cheaper plant, 2. $c\left(y\right) = 2\sqrt{y_2}$.

4b. Question 5.14
Use time series data to estimate marginal cost in each period.

Solution: Take a total derivative of the cost function to get the following:

$$dc = \sum_{i=1}^{n} \frac{\partial c}{\partial w_i} dw_i + \frac{\partial c}{\partial y} dy$$

$$\frac{\partial c}{\partial y} dy = dc - \sum_{i=1}^{n} \frac{\partial c}{\partial w_i} dw_i$$

$$\frac{\partial c}{\partial y} = \frac{1}{dy} \cdot \left[dc - \sum_{i=1}^{n} \frac{\partial c}{\partial w_i} dw_i\right]$$

$$MC = \frac{1}{\Delta y} \cdot \left[\Delta c - \sum_{i=1}^{n} x_i^\star \Delta w_i\right]$$

Using Shepherd's Lemma to find marginal cost as a function of your known variables.

Part I

Short Essay

1 Short Essay 1

1. Your client, a specialty materials supplier, wants to plan how to respond to changes in key input prices (electricity, gadolinium ore, labor) to achieve given levels of output. What sort of data should you gather, and how might you analyze it? Write an overview memo of no more than 250 words that is intelligible to the client's non-technical managers as well as to its data scientists; please print it on a separate page.

Solution: This question is about estimating conditional factor demands, which probably is best done via estimating a cost function and using Shepard's lemma. Credit will be given for sensible ideas on how to do it, and for writing clearly.

2 Short Essay 2

2. What are the differences between (a) increasing returns to scale, (b) learning curve, and (c) decreasing (average) cost? What are the different implications of (a-c) for competitive equilibrium price in the long run? Write about 100 words intelligible to your TA.

Solution:

- Increasing returns to scale is a production function where doubling inputs results in a more than doubling of output
- A learning curve relates to increasing efficiency with experience, and experience is proxied by accumulated output quantity since the firm (or plant) began operation

- Decreasing average cost is related to increasing returns to scale; it implies that marginal cost is below average cost (potentially indefinitely, such as in an industry where fixed costs are the primary costs). This generally occurs because of increasing returns to scale with constant factor prices.

All these things cause problems for CE in the LR; a more likely industry structure in each case is oligopoly or even monopoly.