Problem Set 3

Econ 200

Q1. Your Bernoulli function is $u(m) = \sqrt{m}$, and currently your wealth is m=4. You have a lottery ticket that will pay 12 with probability 0.1 and otherwise will pay 0.
a. What is the mean and variance of the lottery ticket value?

Solution: By definition, the expected value of the lottery ticket is

$$\begin{aligned} E[l] &= px_1 + (1-p)x_2 \\ &= 0.1(12) + 0.9(0) \\ &= 1.2. \end{aligned}$$

By definition, the variance of the lottery ticket is

$$\begin{aligned} V[l] &= p(x_1 - E[l])^2 + (1-p)(x_2 - E[l])^2 \\ &= 0.1(12-1.2)^2 + 0.9(0-1.2)^2 \\ &= 12.96. \end{aligned}$$

b. What is your expected utility with the ticket? Without the ticket?

Solution: With the ticket:
Assuming expected utility hypothesis holds and Bernoulli utility function is given by $u(m) = \sqrt{m}$,

$$\begin{aligned} U(L_w) &= pu(m_{1,w}) + (1-p)u(m_{2,w}) \\ &= 0.1(\sqrt{12+4}) + 0.9(\sqrt{4}) \\ &= 2.2. \end{aligned}$$

Without the ticket:

$$
\begin{aligned}
U(L_{wo}) &= pu(m_{1,wo}) + (1-p)u(m_{2,wo}) \\
&= 1(\sqrt{4}) \\
&= 2.
\end{aligned}
$$

c. What is the maximum price that you rationally would turn down if someone offered to buy your ticket?

Solution: When price received is too low, you turn down the price because in part b), the offered price was 0 and utility is lower in the case without the ticket. Denote the price received by q. The break-even price, q_{BE}, which makes you indifferent between *accept* and *turn down* is

$$
\begin{aligned}
& U(L_w) = U(L_{wo,q_{BE}}) \\
\Leftrightarrow \quad & 2.2 = 1(\sqrt{4 + q_{BE}}) \\
\Leftrightarrow \quad & q_{BE} = 0.84.
\end{aligned}
$$

You will turn down if price offered is $q < 0.84$.

d. What is your risk premium for the lottery ticket?

Solution: By definition, risk premium is the difference between the expected value and Willingness to accept (WTA),

$$
\textit{risk premium} = E[l] - q_{BE} = 1.2 - 0.84 = 0.36.
$$

Even though the expected value of the offer is lower than the expected value of the lottery ticket, you are indifferent between *accept* and *turn down* the offer because the variance is lower for the offer because the offer has variance 0.

Q2. Finance textbooks often assume that utility takes the form $U(L) = E[L] - cVar[L]$, for some $c > 0$ parametrizing the degree of risk aversion. Consider two investors, A and B. Investor A maximizes U[L] for $c = 0.01$, while B maximizes the expected value of the Bernoulli function $u(m) = m^{0.9}$. Consider two lotteries: S pays \$1 for sure, while R pays \$1 with probability 0.999 and pays \$1000 with probability 0.001.

a. Compute the mean and variance of each lottery.

Solution: Lottery S: By definition, the expected value is

$$E[S] = 1(1) = 1,$$

and the variance is

$$V[S] = 1(1-1)^2 = 0.$$

Lottery R: By definition, the expected value is

$$E[R] = 0.999(1) + 0.001(1000) = 1.999,$$

and the variance is

$$V[R] = 0.999(1-1.999)^2 + 0.001(1000-1.999)^2 = 997.003.$$

b. Compute U (for investor A) for each lottery.

Solution: Inverstor A: Using the mean-variance utility, $U(L) = E[L] - cVar[L]$, with $c = 0.01$,

$$U_A(S) = 1 + 0.01(0) = 1,$$
$$U_A(R) = 1.999 - 0.01(997.003) = -7.971.$$

c. Compute Eu (for investor B) for each lottery.

Solution: Inverstor B: Assuming expected utility hypothesis holds and Bernoulli utility function is given by $u(m) = m^{0.9}$,

$$U_B(S) = 1(1^{0.9}) = 1,$$
$$U_B(R) = 0.999(1^{0.9}) + 0.001(1000^{0.9}) = 1.5.$$

d. If faced with the choice between R and S, which would each investor choose?

Solution: The agents choose the lottery which leads to the higher utility. Thus, investor A chooses lottery S and investor B chooses lottery R.

e. Now do Short Essay 1 below.

Solution: See the solution in Essay 1 below.

Q3. A patient has either disease A or disease B. Diagnostic test 1 is positive with probability 0.7 (and negative with probability 0.3) with disease A, and is positive with probability 0.4 with disease B. Similarly, diagnostic test 2 is positive with probability 0.2 when the disease is A, and with probability 0.5 when the disease is B. Overall, the relative incidence of the two diseases is 60% A and 40% B.
a. Compute the joint probabilities p(d, t1, t2) for each disease d=A, B and each test result t1=pos, neg and t2=pos, neg.

Solution: Denote the results of test i as t_i^j, where $j = \{+, -\}$ and "+" means positive. Assume the likelihood functions of test 1 and 2 are independent,

$$p(t_1^{j_1}, t_2^{j_2}|X) = p(t_1^{j_1}|X) \cdot p(t_2^{j_2}|X),$$

where $X = \{A, B\}$ is a disease type. Under this condition, joint probabilities can be calculated as multiplication of likelihood functions and prior probabilities,

$$\begin{aligned}
p(A, t_1^+, t_2^+) &= p(t_1^+, t_2^+|A) \cdot p(A) \\
&= p(t_1^+|A) \cdot p(t_2^+|A) \cdot p(A) = 0.7(0.2)(0.6) = 0.084 \\
p(A, t_1^+, t_2^-) &= p(t_1^+|A) \cdot p(t_2^-|A) \cdot p(A) = 0.7(0.8)(0.6) = 0.336 \\
p(A, t_1^-, t_2^+) &= p(t_1^-|A) \cdot p(t_2^+|A) \cdot p(A) = 0.3(0.2)(0.6) = 0.036 \\
p(A, t_1^-, t_2^-) &= p(t_1^-|A) \cdot p(t_2^-|A) \cdot p(A) = 0.3(0.8)(0.6) = 0.144 \\
p(B, t_1^+, t_2^+) &= p(t_1^+|B) \cdot p(t_2^+|B) \cdot p(B) = 0.4(0.5)(0.4) = 0.08 \\
p(B, t_1^+, t_2^-) &= p(t_1^+|B) \cdot p(t_2^-|B) \cdot p(B) = 0.4(0.5)(0.4) = 0.08 \\
p(B, t_1^-, t_2^+) &= p(t_1^-|B) \cdot p(t_2^+|B) \cdot p(B) = 0.6(0.5)(0.4) = 0.12 \\
p(B, t_1^-, t_2^-) &= p(t_1^-|B) \cdot p(t_2^-|B) \cdot p(B) = 0.6(0.5)(0.4) = 0.12
\end{aligned}$$

b. Compute the prior probabilities of each test result and each disease.

Solution: Prior probabilities of each disease are given in the question.

$$p(A) = 0.6, \quad p(B) = 0.4.$$

Message probabilities are sum of all the combinations of joint probabilities with the message of interest,

$$p(t_1^+) = p(t_1^+|A) \cdot p(A) + p(t_1^+|B) \cdot p(B) = 0.58$$
$$p(t_1^-) = p(t_1^-|A) \cdot p(A) + p(t_1^-|B) \cdot p(B) = 0.42$$
$$p(t_2^+) = p(t_2^+|A) \cdot p(A) + p(t_2^+|B) \cdot p(B) = 0.32$$
$$p(t_2^-) = p(t_2^-|A) \cdot p(A) + p(t_2^-|B) \cdot p(B) = 0.68$$

c. Calculate the posterior probability of disease A when t2 is not performed and t1 is positive (and also t1 negative). Similarly, when t1 is not performed and t2 is positive (and also t2 negative).

Solution: Using Bayes' theorem, posterior probabilities are

$$p(A|t_1^+) = \frac{p(t_1^+|A) \cdot p(A)}{p(t_1^+)} = 0.724$$
$$p(A|t_1^-) = \frac{p(t_1^-|A) \cdot p(A)}{p(t_1^-)} = 0.429$$
$$p(A|t_2^+) = \frac{p(t_2^+|A) \cdot p(A)}{p(t_2^+)} = 0.375$$
$$p(A|t_2^-) = \frac{p(t_2^-|A) \cdot p(A)}{p(t_2^-)} = 0.706$$

d. Calculate and put into a table the posterior probability of disease A for each possible combination of the test results (t1, t2) when both tests are performed. [Hint: you might want to do this problem on a spreadsheet! If so, please show the key cell formulas somewhere on your printout.]

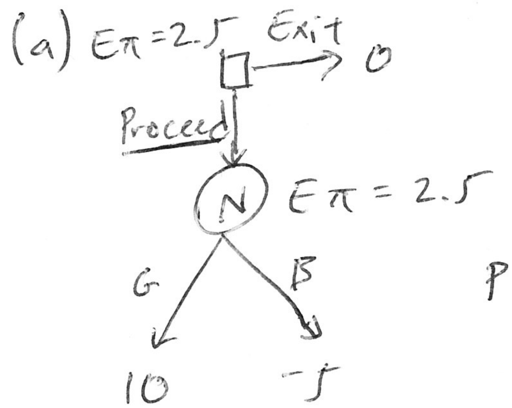
Solution: By Bayes' theorem, posterior probabilities are

$$\begin{aligned}
p(A|t_1^+, t_2^+) &= \frac{p(t_1^+, t_2^+|A) \cdot p(A)}{p(t_1^+, t_2^+)} = \frac{p(t_1^+|A) \cdot p(t_2^+|A) \cdot p(A)}{p(t_1^+, t_2^+)} \\
&= \frac{p(t_1^+|A) \cdot p(t_2^+|A) \cdot p(A)}{p(t_1^+, t_2^+|A) \cdot p(A) + p(t_1^+, t_2^+|B) \cdot p(B)} \\
&= 0.512 \\
p(A|t_1^+, t_2^-) &= 0.808 \\
p(A|t_1^-, t_2^+) &= 0.231 \\
p(A|t_1^-, t_2^-) &= 0.545.
\end{aligned}$$

Q4. If the market for your new product turns out well (G) you will gain incremental profit of 10 (millions of $), but otherwise (B) you will lose 5 if you bring out the product. You can assure an incremental profit of 0 if you cancel now. You believe the probability of G is 0.5. a. Draw the decision tree and solve it; state whether or not you should you bring out the product. b. Suppose a consultant can tell you now whether to expect G or B. Her fee is 3. Is it worthwhile to hire her?

I. 1

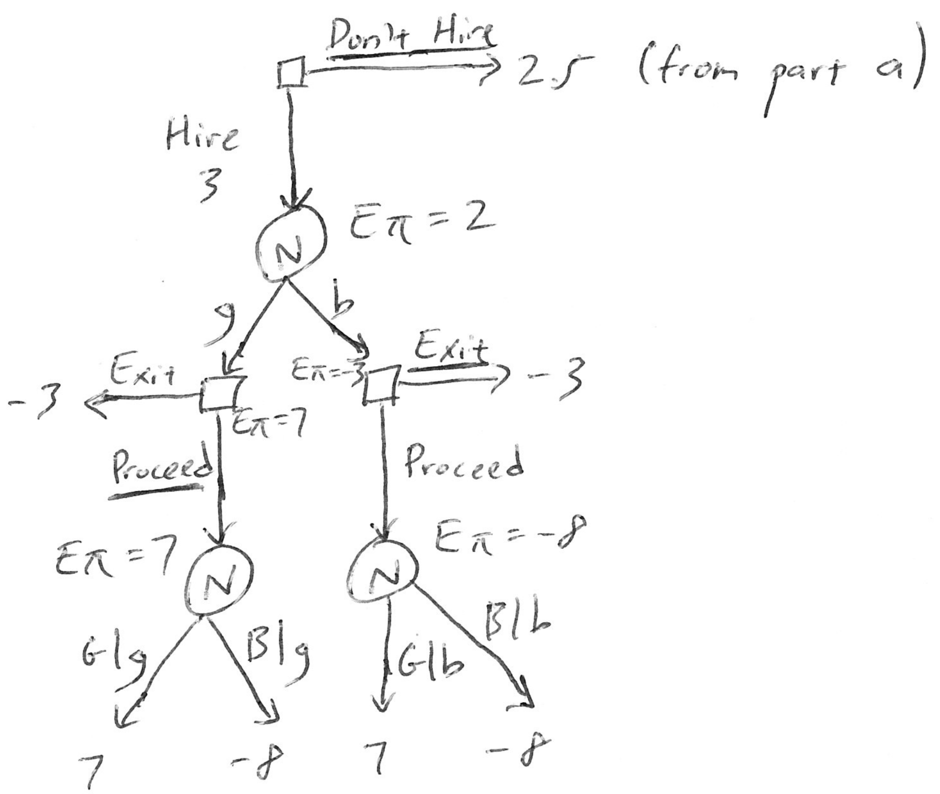
(a)



$P(G) = P(B) = 0.5$

Bring out the product since the expected payoff of doing so (2.5) is greater than exiting (0).

(b) If the consultant is always correct, $P(g|G) = P(b|B) = 1$:



It is not worthwhile to hire the consultant.

Solution: Finite-period dynamic decision problems can be always solved by backward induction. There are two types of nodes, decision nodes and nature nodes. At a decision node, the agent chooses a choice with the highest payoff. At a nature node, we calculate expected value of the lottery.

One important thing is that probability which characterize each lottery must be conditional probability. In part a), conditional probability coincides with unconditional probability because there is no message. However, in part b), in the last node, which is a nature node, we must use conditional probability $p(X|x)$, where $X = \{G, B\}$ is a true state of the nature and $x = \{g, b\}$ is a message because your best guess on the probability of true state is updated once you obtain information from messages. Given prior probability, $p(X)$ and assumptions on likelihood functions, $p(x|X)$, we can solve for posterior probability, $p(X|x)$, as we did in Q3. For example,

$$P(G|g) = \frac{p(g|G)p(G)}{p(g)} = \frac{p(g|G)p(G)}{p(g|G)p(G) + p(g|B)p(B)} = \frac{1(0.5)}{1(0.5) + 0(0.5)} = 1.$$

Also, at the first nature node once you determine to consult, which is a node where the consultant tells you whether a message is *good* or *bad*, we must use message probability, $p(x)$, instead of prior probability, $p(X)$, because message probability is the relevant probability at the node. Under the extreme assumption of perfect accuracy of consultant, i.e. $p(g|G) = p(b|B) = 1$, we obtain $p(x) = p(X)$, but this does not hold in general.

For the current problem, the point is that even if the consultant were perfectly accurate, her advice wouldn't change enough of your actions to make her fee worthwhile — you'd have a .5 chance of avoiding a loss of 5, so the expected gain is only 2.5. More formally, fee $= 3 > 2.5 =$ value of perfect information $>$ value of imperfect information.

Essay 1. In problem 2 above, which investor's choice (if either) makes sense to you? How can you generalize beyond the specific values (e.g., c = 0.01 and the probs and payoffs of R and S) in that problem to make a general point? Write about 100 words intelligible to your TA.

Solution: Lottery R first-order stochastically dominates lottery S, meaning that probability that outcome is higher than given value k is always larger for lottery R for all $k \in \mathbb{R}$. Any rational person would prefer lottery R; it is always at least as good and occasionally better. The given Bernoulli function (or any other Bernoulli function, for that matter) picks this up, and says that R delivers greater expected utility.

The mean-variance approach misses the boat in this case. Mean-variance utility uses a second order Taylor approximation, and the error term can be large when you have a skewed distribution The flaw in the mean-variance model is that it always takes variance to be bad, even when it is upside "risk" which a rational person would welcome.